IPhT Lectures on

Black Hole Perturbation Theory

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Abstract

Some lecture notes on selected topics in black hole perturbation theory.

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1 Introduction

The direct detection of gravitational waves from merging binary systems notoriously marked an important milestone in physics. Much of the success of gravitational-wave astronomy relies on perturbation theory. Perturbative approaches are essential ingredients in the study of gravitational-wave sources in the strong-field regime, such as the ringdown after the merger of a binary system, tidally perturbed compact objects, and extreme mass ratio inspirals. The study of black hole perturbations has a long history, dating back to Regge and Wheeler's work on the odd-parity perturbations of Schwarzschild spacetime in the late 1950s, which was followed up by several other works by Zerilli, Vishveshwara, Press, Chandrasekhar, Detweiler, Teukolsky and others.

These notes cover a short selection of topics in the context of black hole perturbation theory, presented at the 2024 IPhT Lectures in Saclay.[1]

Figure 1: The dynamics of two body systems is usually characterized by three distinct phases: the inspiral (the phase in which the bodies are in a regime of weak gravity and small velocities), the merger and the ringdown (the final stage when object is relaxing to its equilibrium configuration). Standard techniques rely on perturbative methods to model the evolution of inspiral and ringdown, while numerical relativity computations are used to resolve the nonlinear intermediate stage of the merger.

Conventions: Throughout we use the mostly-plus metric signature, and work in natural units $\hbar = c = 1$. We also often set $G = 1$. We will often work in generic D-spacetime dimensions. We denote spacetime indices using Greek letters, e.g., $\mu, \nu, \rho, \cdots$, denote spatial indices by Latin letters from the beginning of the alphabet, e.g., $a, b, c, \cdots$, and we denote angular indices on the S^{D-2}-sphere using Latin indices from the middle of the alphabet, e.g., $i, j, k, \cdots$. We will be using the following convention for (anti-)symmetrization of indices: $A_{(\mu\nu)} \equiv \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$, $B_{[\mu\nu]} \equiv \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$. $(\cdots)_T$ will denote the symmetrized traceless component of the enclosed indices.

[1] See https://courses.ipht.fr/?q=en/all-courses for a list of courses.

2 A black hole perturbation theory primer

We begin by reviewing the main aspects of black hole perturbation theory in the simpler case of non-rotating black holes. Although the main focus of the lecture notes will be on black holes in four-dimensional general relativity, it is instructive to set up the problem in generic D-spacetime dimension. Results in four dimensions can be obtained by simply setting $D = 4$ in the expressions below.[2] This is useful because, as we shall see, although the formulas below are valid in any $D \geq 4$, some properties of the dynamics of the perturbations, including duality symmetries, are exclusive of $D = 4$ and do not hold in $D > 4$. We will comment on these particular aspects below.

In the four-dimensional context, the linearized dynamics of the perturbations around an asymptotically flat Schwarzschild black hole was first worked out by Regge and Wheeler [1] and by Zerilli [2, 3], and in a gauge-invariant formalism by Moncrief, Cunningham, and Price [4–6]. In higher dimensions, the equations of motion for these perturbations can be found in [7–10]. (We will be mostly following [10] here.) For the generalization to massive and partially massless spin-2 fields on a Schwarzschild–(A)dS spacetime in D-dimensions, see e.g. [11].

2.1 Perturbations of Schwarzschild black holes

The Schwarzschild–(anti-)de Sitter (S(A)dS) geometry $\bar{g}_{\mu\nu}$ in D spacetime dimensions is described by the line element

$$\mathrm{d}s^2 = \bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{S^{D-2}}, \tag{2.1}$$

where

$$f(r) = 1 - \left(\frac{r_s}{r}\right)^{D-3} - \frac{2\Lambda r^2}{(D-1)(D-2)}, \tag{2.2}$$

and where $\mathrm{d}\Omega^2_{D-2}$ is the line element on the $(D-2)$-sphere, defined recursively by

$$\mathrm{d}\Omega^2_{S^n} = \mathrm{d}\theta_n^2 + \sin^2\theta_n \mathrm{d}\Omega^2_{S^{n-1}}, \qquad \text{with } \mathrm{d}\Omega^2_{S^1} = \mathrm{d}\theta_1^2, \tag{2.3}$$

where the coordinate θ_1 ranges from 0 to 2π, whereas all other angles θ_i range from 0 to π. The parameter Λ corresponds to the cosmological constant (the geometry corresponds to Schwarzschild–anti-de Sitter for negative Λ, and Schwarzschild–de Sitter for positive Λ), while r_s denotes the Schwarzschild radius of the asymptotically flat black hole in the limit $\Lambda = 0$. It is related to the asymptotically flat black hole mass via

$$GM = \frac{(D-2)}{16\pi}\left(\frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left[\frac{D-1}{2}\right]}\right) r_s^{D-3}. \tag{2.4}$$

In many cases it will be convenient to introduce the so-called tortoise radial coordinate, defined through

$$\mathrm{d}r_\star = \frac{1}{f(r)}\mathrm{d}r. \tag{2.5}$$

[2] Alternatively, a reader that is interested uniquely on the $D = 4$ case can jump directly to the box below where the original results by Regge and Wheeler, and Zerilli are summarized.

The S(A)dS spacetime is both static and rotationally symmetric. It possesses $(D-1)(D-2)/2+1$ isometries which form the group $\mathbb{R}\times \mathrm{SO}(D-1)$. One of these isometries corresponds to time translations, reflecting the time independence of the geometry, while the rest are the rotational symmetries of surfaces of constant radius at fixed time.[3]

We are interested in studying perturbations around the background geometry $\bar{g}_{\mu\nu}$. Therefore, we shall perturb $\bar{g}_{\mu\nu}$ as $\bar{g}_{\mu\nu}+h_{\mu\nu}$ and study the dynamics of the fluctuation $h_{\mu\nu}$ in perturbation theory. In order to maximally utilize the $\mathrm{SO}(D-1)$ symmetry, it is useful to decompose fields propagating in the spacetime into spherical harmonics. In particular, we will decompose $h_{\mu\nu}$ into scalar, vector, and tensor spherical harmonics. This is convenient because rotational invariance of the background guarantees that modes associated to different types of spherical harmonics decouple from each other at linear order. To provide an analogy, this is reminiscent of the scalar, vector and tensor decomposition on an FLRW spacetime in cosmology: in a similar way, homogeneity and isotropy of the FLRW background ensure that different modes decouple at the level of the linearized equations of motion.

We shall thus split $h_{\mu\nu}$ as follows:

$$
\begin{aligned}
h_{\mu\nu} = \sum_{L,M} &\begin{pmatrix} f(r)H_0(t,r) & H_1(t,r) & \mathcal{H}_0(t,r)\nabla_i \\ * & f(r)^{-1}H_2(t,r) & \mathcal{H}_1(t,r)\nabla_i \\ * & * & r^2\left[\mathcal{K}(t,r)\gamma_{ij}+G(t,r)\nabla_{(i}\nabla_{j)_T}\right]\end{pmatrix} Y_L^M \\
&+\sum_{L,M}\begin{pmatrix} 0 & 0 & h_0(t,r)Y_i^{(T)}{}_L^M \\ * & 0 & h_1(t,r)Y_i^{(T)}{}_L^M \\ * & * & r^2h_2(t,r)\nabla_{(i}Y_{j)}^{(T)}{}_L^M\end{pmatrix} + \sum_{L,M}\begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & r^2h_T(t,r)\end{pmatrix} Y_{ij}^{(TT)}{}_L^M, \qquad (2.6)
\end{aligned}
$$

where asterisks correspond to symmetric components, $(\cdots)_T$ denotes the trace-free symmetrized part of the enclosed indices, and ∇_i denotes the covariant derivative on the S^n sphere.[4] Y are the scalar spherical harmonics, $Y_i^{(T)}$ are the (transverse) vector harmonics, and $Y_{ij}^{(TT)}$ are the (transverse and traceless) tensor harmonics; they are orthogonal to each other and satisfy the following eigenvalue equations [10, 13]:

$$\Delta_{S^n}Y_L(\theta) = -L(L+n-1)Y_L(\theta), \tag{2.7}$$

$$\Delta_{S^n}Y_i^{(T)}{}_L^M = -\left(L(L+n-1)-1\right)Y_i^{(T)}{}_L^M, \tag{2.8}$$

$$\Delta_{S^n}Y_{ij}^{(TT)}{}_L^M = -\left(L(L+n-1)-2\right)Y_{ij}^{(TT)}{}_L^M, \tag{2.9}$$

where $n\equiv D-2$ and Δ_{S^n} is the laplacian on the S^n sphere, which can be recursively expressed as

$$\Delta_{S^n} = \sin^{1-n}\theta_n\frac{\partial}{\partial\theta_n}\sin^{n-1}\theta_n\frac{\partial}{\partial\theta_n}+\sin^{-2}\theta_n\Delta_{S^{n-1}}. \tag{2.10}$$

Further details can be found, e.g., in appendix A of [10].

[3] In addition to these continuous symmetries, the S(A)dS background is invariant under discrete parity transformations, which are typically taken to act by mapping points on a constant-radius sphere to their antipode.

[4] The decomposition into spherical harmonics obscures the counting of propagating degrees of freedom, which for a massless spin-2 field on flat D-dimensional space is $D(D-3)/2$. The counting can however be recovered by matching onto a plane wave basis, see e.g. [12].

There are correspondingly three different sectors of perturbations: one consists of the perturbations proportional to (derivatives of) scalar harmonics:

$$H_0, \quad H_1, \quad H_2, \quad \mathcal{H}_0, \quad \mathcal{H}_1, \quad G, \quad \mathcal{K} \qquad\qquad \text{(scalar perturbations)}, \tag{2.11}$$

which we will refer to as scalar perturbations for simplicity. Another sector includes perturbations proportional to (derivatives of) vector harmonics:

$$h_0, \quad h_1, \quad h_2 \qquad\qquad \text{(vector perturbations)}. \tag{2.12}$$

Finally there is a single perturbation proportional to a tensor harmonic:

$$h_T, \qquad\qquad \text{(tensor perturbation)}. \tag{2.13}$$

Since these perturbations all multiply different kinds of spherical harmonics (which have different SO$(D-1)$ Casimir eigenvalues), the three sectors decouple at the linear level.

Note that in $D=4$ the h_T perturbation is absent: the most general decomposition of a tensor harmonic is contained already in the building blocks $\mathcal{K}(t,r)\gamma_{ij}Y_L^M$, $G(t,r)\nabla_{(i}\nabla_{j)_T}Y_L^M$ and $h_2(t,r)\nabla_{(i}Y^{(T)}_{j)}{}^M_L$ [1]. In this case the eigenvalues of the scalar and vector spherical harmonics happen to coincide for the two-sphere but the scalar and vector modes continue to decouple because they have different parity eigenvalues. In this case, the scalar (vector) modes correspond to the usual even (odd) modes.

Note also that in $D=4$ we can replace the unit-normalized vector spherical harmonic in terms of a gradient of a scalar harmonic as $Y_i^{(T)}{}_L^M \xrightarrow{D\to 4} \epsilon^j{}_i\nabla_j Y_L^M/\sqrt{L(L+1)}$.[5]

Gauge transformations. A massless spin-2 field enjoys gauge invariance under linearized diffeomorphisms

$$\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{2.14}$$

In order to see how the various fields shift under this transformation, we split the diffeomorphism parameter, ξ in a similar way to what we did for the spin-1 field:

$$\xi_t = \sum_{L,M} f(r)\xi_0(t,r)Y_L^M, \tag{2.15a}$$

$$\xi_r = \sum_{L,M} f^{-1}\xi_1(t,r)Y_L^M, \tag{2.15b}$$

$$\xi_i = \sum_{L,M} \xi_S(t,r)\nabla_i Y_L^M + \xi_V(t,r)Y_i^{(T)}{}_L^M. \tag{2.15c}$$

[5] There is a small difference in the normalization factor here with respect to the standard literature [1]. In addition, note that there are other small differences between (2.6) and the definition of the metric perturbations in [1]. For instance, the definitions of $\mathcal{K}$ and G differ by the subtraction of the trace from the harmonic multiplying G here, which was not done in [1].

From this we can determine how the individual variables shift under a diffeomorphism:

$$\delta H_0 = 2\dot{\xi}_0 - \frac{f'}{f}\xi_1, \qquad \delta h_0 = \dot{\xi}_V, \qquad \delta h_T = 0, \tag{2.16a}$$

$$\delta H_1 = f^{-1}\dot{\xi}_1 + f\xi_0', \qquad \delta h_1 = -2r^{-1}\xi_V + \xi_V', \tag{2.16b}$$

$$\delta H_2 = 2\xi_1' - \frac{f'}{f}\xi_1, \qquad \delta h_2 = 2r^{-2}\xi_V, \tag{2.16c}$$

$$\delta \mathcal{H}_0 = \dot{\xi}_S + f\xi_0, \tag{2.16d}$$

$$\delta \mathcal{H}_1 = f^{-1}\xi_1 - 2r^{-1}\xi_S, \tag{2.16e}$$

$$\delta \mathcal{K} = 2r^{-1}\xi_1 - \frac{2L(L+D-3)}{(D-2)r^2}\xi_S, \tag{2.16f}$$

$$\delta G = 2r^{-2}\xi_S. \tag{2.16g}$$

Notice that the mode multiplying the tensor spherical harmonic $Y_{ij}^{(TT)}$ is gauge invariant.

Quadratic action for the perturbations. Our starting point is the Einstein–Hilbert action[6]

$$S = \int \mathrm{d}^D x \sqrt{-g}\, \frac{M_{\rm Pl}^{D-2}}{2}(R - 2\Lambda), \tag{2.17}$$

which we will expand up to quadratic order in perturbations. Since perturbations belonging to the scalar, vector and tensor sectors decouple at this order, we shall treat them separately below.

Vector sector. Let's start from the vector sector. The variables in this sector are h_0, h_1, h_2. However, we expect that there should only be one physical combination of these degrees of freedom, so we will have to fix a gauge and integrate out auxiliary variables.

We choose to work in the so-called Regge–Wheeler gauge [1], defined by the condition

$$h_2 = 0. \tag{2.18}$$

It is clear from (2.16d) that we have enough freedom to reach this gauge by choosing ξ_V appropriately. Note that fixing $h_2 = 0$ directly at the level of the action is a consistent choice since the Regge–Wheeler gauge is a full gauge fixing [14]. The remaining degrees of freedom are h_0 and h_1, and their action is given (up to integrations by parts) by

$$\begin{aligned} S_{\rm RW} = \int \mathrm{d}t\mathrm{d}r\, r^{D-4}\bigg[&\dot{h}_1^2 + {h_0'}^2 + \frac{4}{r}h_0\dot{h}_1 - 2h_0'\dot{h}_1 + \frac{2(D-3)f + 2rf' - (L+1)(D-4+L)}{r^2} f\, h_1^2 \\ &+ \frac{(L+1)(D-4+L) - 2rf'}{r^2 f} h_0^2 + \frac{4\Lambda}{D-2}\left(f h_1^2 - f^{-1}h_0^2\right)\bigg]. \end{aligned} \tag{2.19}$$

In this choice of variables, it is somewhat difficult to isolate the physical degree of freedom because neither h_0 nor h_1 is obviously auxiliary. It is therefore useful to integrate in an additional auxiliary field Q in a

[6] Recall that the canonically normalized fluctuation for the metric is $g_{\mu\nu} = \bar{g}_{\mu\nu} + 2h_{\mu\nu}/M_{\rm Pl}^{(D-2)/2}$, with $\bar{g}_{\mu\nu}$ the Einstein background metric.

similar manner to [15–17], so that our action becomes

$$S_{\rm RW} = \int {\rm d}t{\rm d}r\, r^{D-4}\bigg[2Q\left(\dot h_1 + \frac{2}{r}h_0 - h_0'\right) - Q^2 + \frac{2(D-3)f + 2rf' - (L+1)(D-4+L)}{r^2}f\, h_1^2 \\ + \frac{(L+1)(D-4+L) - 2(D-3)f - 2rf'}{r^2 f}h_0^2 + \frac{4\Lambda}{D-2}\left(fh_1^2 - f^{-1}h_0^2\right)\bigg]. \tag{2.20}$$

The organizing principle is to make the derivatives of h_0 and h_1 appear as a perfect square and then to introduce Q in such a way that integrating it out reproduces the original action.

The actions (2.20) and (2.19) are equivalent, but we can now integrate out h_0 and h_1 to get an action only for Q. Their equations of motion set

$$h_0 = -\frac{rf}{(L-1)(D-2+L)}\left[(D-2)Q + rQ'\right], \tag{2.21}$$

$$h_1 = -\frac{r^2}{(L-1)(D-2+L)f}\dot Q. \tag{2.22}$$

We can then substitute these equations back into the action. In simplifying the resulting expression, it is helpful to use the background equations of motion, which imply

$$f'' + \frac{(D-2)f'}{r} + \frac{4\Lambda}{D-2} = 0. \tag{2.23}$$

The action for Q then takes the form (after several integrations by parts)

$$S_{\rm RW} = \int {\rm d}t{\rm d}r \frac{r^{D-2}}{(L-1)(D-2+L)}\left[\frac{1}{f}\dot Q^2 - fQ'^2 - \left(\frac{(L+1)(D-4+L) - (D-4)f}{r^2} - \frac{Df'}{r} - \frac{4\Lambda}{D-2}\right)Q^2\right]. \tag{2.24}$$

It is again useful to write things in terms of a canonically normalized Schrödinger variable. We define

$$\Psi_{\rm RW} \equiv \left(\frac{2r^{D-2}}{(L-1)(D-2+L)}\right)^{1/2} Q, \tag{2.25}$$

and transform to the tortoise coordinate. After integrations by parts the action takes the form

$$S_{\rm RW} = \int {\rm d}t{\rm d}r_\star \left(\frac{1}{2}\dot\Psi_{\rm RW}^2 - \frac{1}{2}\left(\frac{\partial \Psi_{\rm RW}}{\partial r_\star}\right)^2 - \frac{1}{2}V_{\rm RW}(r)\Psi_{\rm RW}^2\right), \tag{2.26}$$

with the Regge–Wheeler potential

$$V_{\rm RW}(r) = f\frac{(L+1)(D-4+L)}{r^2} + f^2\frac{(D-4)(D-6)}{4r^2} - ff'\frac{(D+2)}{2r} - \frac{4\Lambda f}{D-2}. \tag{2.27}$$

The Schrödinger equation following from the action (2.26) is

$$\frac{{\rm d}^2\Psi_{\rm RW}}{{\rm d}r_\star^2} + \Big(\omega^2 - V_{\rm RW}(r)\Big)\Psi_{\rm RW} = 0, \tag{2.28}$$

which is precisely the D-dimensional Regge–Wheeler equation [8]. Note, however, that $\Psi_{\rm RW}$ is not quite the usual Regge–Wheeler variable: it is more properly called the Cunningham–Price–Moncrief variable [5, 18].

The usual Regge–Wheeler variable is (up to a numerical factor) the time derivative of what we have called $\Psi_{\rm RW}$—see the box below and eq. (2.60) for a comparison with the Regge–Wheeler derivation in $D = 4$. However, since both of these variables satisfy the same (Regge–Wheeler) equation, we will slightly abuse terminology and refer to $\Psi_{\rm RW}$ as the Regge–Wheeler variable.

Scalar sector. Let's then consider the scalar sector. In $D = 4$, this coincides with the even parity sector. In this case, the relevant degrees of freedom are the variables $H_0, H_1, H_2, \mathcal{H}_0, \mathcal{H}_1, G, \mathcal{K}$.

It is convenient to fix a gauge where[7]

$$\mathcal{H}_0 = \mathcal{K} = G = 0, \tag{2.29}$$

so that the residual degrees of freedom are $H_0, H_1, H_2, \mathcal{H}_1$. With the gauge choice (2.29), the action becomes

$$\begin{aligned} S_{\rm Z} = L(L+D-3)\int {\rm d}t{\rm d}r\, r^{D-4}\Bigg[&\dot{\mathcal{H}}_1^2 + \frac{2f}{r^2}\left((D-3)f + \frac{2\Lambda r^2}{D-2} + rf'\right)\mathcal{H}_1^2 + \frac{(D-2)(rf'+(D-3)f)}{2L(L+D-3)}H_2^2 \\ &+ H_0\left(\left[f' + \frac{2(D-3)f}{r}\right]\mathcal{H}_1 + 2f\mathcal{H}_1' - \left(1 + \frac{(D-3)(D-2)f+(D-2)rf'}{L(L+D-3)}\right)H_2 - \frac{(D-2)rfH_2'}{L(L+D-3)}\right) \\ &- \frac{(2(D-3)f+rf')}{r}H_2\mathcal{H}_1 + H_1\left(H_1 - 2\dot{\mathcal{H}}_1 + \frac{2(D-2)r}{L(L+D-3)}\dot{H}_2\right)\Bigg]. \end{aligned} \tag{2.30}$$

Though this expression is fairly complicated, we expect that there should be a single physical degree of freedom, and we would like to isolate it.

It is reasonably clear from the action that H_1 is auxiliary; it can be integrated out via its equation of motion, which sets

$$H_1 = \dot{\mathcal{H}}_1 - \frac{(D-2)r}{L(L+D-3)}\dot{H}_2. \tag{2.31}$$

The variable H_2 is also auxiliary, but in order to integrate it out we first trade off $\mathcal{H}_1$ for another variable $\mathcal{V}$ defined by

$$\mathcal{V} = \mathcal{H}_1 - \frac{(D-2)r}{2L(L+D-3)}H_2. \tag{2.32}$$

This makes H_2 appear algebraically in the H_0 equation of motion (which is a constraint), so that it can be solved for in terms of $\mathcal{V}$:

$$H_2 = \frac{2L(L+D-3)\Big[2f(r\mathcal{V}' + (D-3)\mathcal{V}) + rf'\mathcal{V}\Big]}{r\Big[2L(L+D-3) - 2(D-2)f + (D-2)rf'\Big]}. \tag{2.33}$$

Substituting this solution back into the action eliminates both H_2 and H_0, because we have solved the constraint that the latter enforces. We are therefore left with an action for only the variable $\mathcal{V}$. After

[7] Note that this is slightly different compared to the Regge–Wheeler gauge of [1]—see also the box below. Analogously to the vector sector above, both this gauge and the Regge–Wheeler gauge in the vector/odd sector can be used in the action, as shown explicitly in [14], without losing information contained in the equations of motion.

integration by parts it can be written as

$$S_{\rm Z} = \int {\rm d}t{\rm d}r\, r^{D-4}\mathcal{F}(r)\left(\dot{\mathcal{V}}^2 - f^2\mathcal{V}'^2 + \mathcal{N}(r)\mathcal{V}^2\right)\,, \tag{2.34}$$

where the functions that appear are rather complicated:

$$\mathcal{F}(r) \equiv \frac{8(D-2)L(L+D-3)f\Big[(D-3)L(L+D-3)-(D-3)(D-2)f-2\Lambda r^2-(D-2)rf'\Big]}{\Big[2L(L+D-3)-2(D-2)f+(D-2)rf'\Big]^2}, \tag{2.35a}$$

$$\mathcal{N}(r) \equiv \frac{\mathcal{A}+\mathcal{B}+\mathcal{C}}{r^2(2L(L+D-3)^2-2(D-2)f+(D-2)rf')}, \tag{2.35b}$$

with the following expressions appearing in $\mathcal{N}$:

$$\mathcal{A} = 2f\left[-L^2(L+D-3)^2 + f\left(2L(L+D-3)+(D-4)(D-2)f+4(D-3)\Lambda r^2\right)\right], \tag{2.36a}$$

$$\mathcal{B} = rff'\left[3(D-4)L(L+D-3)+(D-2)(24+D(2D-13))f+4\Lambda r^2\right], \tag{2.36b}$$

$$\mathcal{C} = r^2f'^2\left[2L(L+D-3)+(D-2)(3D-10)f+(D-2)rf'\right]. \tag{2.36c}$$

We can canonically normalize the action by defining the Zerilli variable

$$\Psi_Z \equiv \left(2fr^{D-4}\mathcal{F}\right)^{1/2}\mathcal{V}, \tag{2.37}$$

to finally obtain

$$S_{\rm Z} = \int {\rm d}t{\rm d}r_\star\left(\frac{1}{2}\dot{\Psi}_{\rm Z}^2 - \frac{1}{2}\left(\frac{\partial\Psi_{\rm Z}}{\partial r_\star}\right)^2 - \frac{1}{2}V_{\rm Z}(r)\Psi_{\rm Z}^2\right), \tag{2.38}$$

with the Zerilli potential

$$V_{\rm Z}(r) = \frac{f\,\hat{V}_Z(r)}{4(D-2)r^2\bar{H}(r)^2}, \tag{2.39}$$

where we have defined the functions

$$\bar{H}(r) \equiv 2L(L+D-3)-2(D-2)f+(D-2)rf' \tag{2.40}$$

$$\begin{aligned}\hat{V}_Z(r) \equiv\ & 4(D-4)(D-2)^4f^3 - 8(D-2)^2\Big[(D-2)(D-6)L(L+D-3)-8(D-3)\Lambda r^2\Big]f^2\\ &+4(D-2)\Big[(D-2)(D-12)L^2(L+D-3)^2-16(D-4)L(L+D-3)\Lambda r^2+32\Lambda^2r^4\Big]f\\ &+2(D-2)^3(D+2)r^3f'^3-4(D-2)^2r^2\Big[(D-6)L(L+D-3)-4\Lambda r^2\Big]f'^2\\ &-8(D-2)^2L^2(L+D-3)^2rf'+12(D-2)^5rf^2f'+(D-2)^3(D(D+10)-32)r^2ff'^2\\ &-4(D-2)^2\Big[(D-2)(3D-8)L(L+D-3)-8D\Lambda r^2\Big]rff'\\ &+16L^2(L+D-3)^2\Big[(D-2)L(L+D-3)-4\Lambda r^2\Big].\end{aligned} \tag{2.41}$$

The corresponding equation of motion

$$\frac{\mathrm{d}^2\Psi_{\mathrm{Z}}}{\mathrm{d}r_\star^2}+\Big(\omega^2-V_{\mathrm{Z}}(r)\Big)\Psi_{\mathrm{Z}}=0, \tag{2.42}$$

is the D-dimensional Zerilli equation, and agrees with the expression derived in [8]. In $D=4$ with $\Lambda=0$, it agrees with the usual Zerilli variable [18].

Black hole perturbations in $D=4$. The notation and gauge choice in the derivation above do not exactly mirror the original works by Regge and Wheeler [1], and Zerilli [2]. For completeness, we will review here how the equations for the even and odd perturbations in $D=4$ were originally derived. The Schwarzschild metric in four-dimensional spacetime is given by the line element

$$\mathrm{d}s^2=-f(r)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{f(r)}+r^2\mathrm{d}\Omega_{S^2}^2\,,\qquad f(r)=1-\frac{r_s}{r}, \tag{2.43}$$

where $\mathrm{d}\Omega_{S^2}$ is the line element on the 2-sphere, $\mathrm{d}\Omega_{S^2}^2=\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2$. We shall decompose the metric fluctuation as [1]

$$\delta g_{\mu\nu}=\delta g_{\mu\nu}^{\mathrm{odd}}+\delta g_{\mu\nu}^{\mathrm{even}}\,, \tag{2.44}$$

with

$$\delta g_{\mu\nu}^{\mathrm{odd}}=\begin{pmatrix}0 & 0 & \varepsilon^k{}_j\nabla_k h_0\\ 0 & 0 & \varepsilon^k{}_j\nabla_k h_1\\ \varepsilon^k{}_i\nabla_k h_0 & \varepsilon^k{}_i\nabla_k h_1 & \frac{1}{2}(\varepsilon_i{}^k\nabla_k\nabla_j+\varepsilon_j{}^k\nabla_k\nabla_i)h_2\end{pmatrix} \tag{2.45}$$

which parametrizes parity-odd perturbations (indices are raised and lowered with $\gamma_{ij}\equiv\mathrm{diag}(1,\sin^2\theta)$), and

$$\delta g_{\mu\nu}^{\mathrm{even}}=\begin{pmatrix}f(r)H_0 & H_1 & \nabla_j\mathcal{H}_0\\ H_1 & H_2/f(r) & \nabla_j\mathcal{H}_1\\ \nabla_i\mathcal{H}_0 & \nabla_i\mathcal{H}_1 & C(\mathcal{K}\gamma_{ij}+\nabla_i\nabla_j G)\end{pmatrix} \tag{2.46}$$

for perturbations of the even type. ∇_i denotes a covariant derivative on the 2-sphere S^2, and in standard coordinates with metric $\gamma_{ij}\equiv\mathrm{diag}(1,\sin^2\theta)$ it follows that

$$\begin{aligned}&\nabla_\theta\nabla_\theta=\partial_\theta^2\,,\qquad \nabla_\phi\nabla_\phi=\partial_\phi^2+\sin\theta\cos\theta\partial_\theta\,,\quad \nabla_\theta\nabla_\phi=\nabla_\phi\nabla_\theta=\partial_\theta\partial_\phi-\frac{\cos\theta}{\sin\theta}\partial_\phi\\ &\nabla^2=\partial_\theta^2+\frac{1}{\sin^2\theta}\partial_\phi^2+\frac{\cos\theta}{\sin\theta}\partial_\theta\,.\end{aligned} \tag{2.47}$$

The Levi–Civita tensors are, in components,

$$\begin{aligned}&\begin{pmatrix}\varepsilon_{\theta\theta} & \varepsilon_{\theta\phi}\\ \varepsilon_{\phi\theta} & \varepsilon_{\phi\phi}\end{pmatrix}=\sin\theta\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\quad,\quad \begin{pmatrix}\varepsilon^\theta{}_\theta & \varepsilon^\theta{}_\phi\\ \varepsilon^\phi{}_\theta & \varepsilon^\phi{}_\phi\end{pmatrix}=\begin{pmatrix}0 & \sin\theta\\ -1/\sin\theta & 0\end{pmatrix}\\ &\begin{pmatrix}\varepsilon_\theta{}^\theta & \varepsilon_\theta{}^\phi\\ \varepsilon_\phi{}^\theta & \varepsilon_\phi{}^\phi\end{pmatrix}=\begin{pmatrix}0 & 1/\sin\theta\\ -\sin\theta & 0\end{pmatrix}\quad,\quad \begin{pmatrix}\varepsilon^{\theta\theta} & \varepsilon^{\theta\phi}\\ \varepsilon^{\phi\theta} & \varepsilon^{\phi\phi}\end{pmatrix}=\frac{1}{\sin\theta}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.\end{aligned} \tag{2.48}$$

The metric components h_0, h_1 and h_2 are pseudo-scalars, i.e. they flip sign under a parity transformation $(\theta,\phi)\to(\pi-\theta,\phi+\pi)$, whereas the other scalar components do not. Let's choose the gauge

$$\mathcal{H}_0=\mathcal{H}_1=G=h_2=0\qquad\qquad\text{(Regge–Wheeler gauge)}, \tag{2.49}$$

and decompose in spherical harmonics as, e.g., $H_0 = \sum_{LM} \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-i\omega t}\, H_0^{(L,M)}(r) Y_L^M(\theta,\varphi)$. Plugging the decomposition (2.44) into the Einstein equations, one then finds the constraint $H_0 = H_2$ and obtains the following first-order differential equations for the even sector [2]

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}r} + \frac{2r-3r_s}{2r(r-r_s)}\mathcal{K} - \frac{1}{r}H_2 + \frac{1}{2}\frac{L(L+1)}{i\omega r^2}H_1 = 0, \tag{2.50}$$

$$\frac{\mathrm{d}H_2}{\mathrm{d}r} - \frac{2r_s - r}{r(r_s-r)}H_2 - \frac{2r-3r_s}{2r(r_s-r)}\mathcal{K} + \left[\frac{i\omega r}{r-r_s} + \frac{L(L+1)}{2i\omega r^2}\right]H_1 = 0, \tag{2.51}$$

$$\frac{\mathrm{d}H_1}{\mathrm{d}r} - \frac{r_s}{r(r_s-r)}H_1 - \frac{i\omega r}{r_s-r}H_2 - \frac{i\omega r}{r_s-r}\mathcal{K} = 0, \tag{2.52}$$

and the algebraic identity

$$\frac{L(L+1)r_s - 4r^3\omega^2}{2i\omega r^2}H_1 - \frac{\left(L^2+L-2\right)r + 3r_s}{r}H_2 + \frac{2\left(L^2+L-3\right)r_s r - 2r^2\left(L^2+L-2r^2\omega^2-2\right) + 3r_s^2}{2r(r_s-r)}\mathcal{K} = 0. \tag{2.53}$$

One can easily show that, after the field redefinitions

$$\mathcal{K} = f(r)\frac{\mathrm{d}\Psi_{\mathrm{Z}}}{\mathrm{d}r} + \frac{3r_s^2 + 3\lambda r_s r + 2(\lambda+1)\lambda r^2}{r^2(3r_s+2\lambda r)}\Psi_{\mathrm{Z}}, \tag{2.54}$$

$$H_1 = -f(r)\frac{i\omega r^2}{r-r_s}\frac{\mathrm{d}\Psi_{\mathrm{Z}}}{\mathrm{d}r} + \frac{i\omega\left(3r_s^2 + 6\lambda r_s r - 4\lambda r^2\right)}{2(r-r_s)(3r_s+2\lambda r)}\Psi_{\mathrm{Z}}, \tag{2.55}$$

with $\lambda \equiv \frac{1}{2}(L-1)(L+2)$, and where H_2 has been fixed from (2.53), the Einstein equations boil down to the single second-order, Schrödinger-like, Zerilli equation (2.42) in $D=4$, with potential

$$V_{\mathrm{Z}}(r) = f(r)\left(\frac{r_s}{r^3} + \frac{2\lambda}{3r^2} + \frac{8\lambda^2(2\lambda+3)}{3(2\lambda r + 3r_s)^2}\right). \tag{2.56}$$

For the odd sector, the Einstein equations reduce to [1]

$$\frac{\mathrm{d}^2 h_0}{\mathrm{d}r^2} + i\omega\frac{\mathrm{d}h_1}{\mathrm{d}r} - \frac{2f(r) + L^2 + L - 2}{r^2 f(r)}h_0 + \frac{2i\omega}{r}h_1 = 0, \tag{2.57}$$

$$\frac{\mathrm{d}h_0}{\mathrm{d}r} - \frac{2}{r}h_0 + \left(i\omega + \frac{\left(L^2+L-2\right)f(r)}{i\omega r^2}\right)h_1 = 0, \tag{2.58}$$

$$\frac{\mathrm{d}h_1}{\mathrm{d}r} + \frac{i\omega}{f(r)^2}h_0 + \left(\frac{1}{rf(r)} - \frac{1}{r}\right)h_1 = 0. \tag{2.59}$$

The first equation above is a consequence of the last two. One can then use the third equation to solve algebraically for h_0. Plugging the solution back into the second equation and defining

$$\Psi_{\mathrm{RW}} \equiv \frac{f(r)}{r}h_1, \tag{2.60}$$

one finds the Schrödinger-like, Regge–Wheeler equation (2.28) with potential (2.27) with $D=4$:

$$V_{\mathrm{RW}}(r) = \left(1 - \frac{r_s}{r}\right)\left(\frac{L(L+1)}{r^2} - \frac{3r_s}{r^3}\right). \tag{2.61}$$

Tensor sector. Let's conclude with the tensor sector. Inserting the decomposition (2.6) into the linearized Einstein–Hilbert action, we get for the h_T mode (up to total derivatives):

$$S_T = \int \mathrm{d}t\mathrm{d}r\, r^{D-2}\left(\frac{1}{2f}\dot{h}_T^2 - \frac{1}{2}fh_T'^2 + f\left[\frac{D-3}{r^2} + \frac{6-2D-L(L+D-3)}{2fr^2} + \frac{f'}{rf} + \frac{2\Lambda}{(D-2)f}\right]h_T^2\right). \tag{2.62}$$

The equation of motion following from this action was derived in [8, 19]. We can express this action in terms of a canonically normalized variable by making the field redefinition

$$\Psi_T \equiv r^{\frac{D-2}{2}} h_T, \tag{2.63}$$

and adopting the tortoise coordinate $\mathrm{d}r_\star = f^{-1}\mathrm{d}r$. The action for Ψ_T is then (once again, after some integration by parts)

$$S_T = \int \mathrm{d}t\mathrm{d}r_\star\left(\frac{1}{2}\dot{\Psi}_T^2 - \frac{1}{2}\left(\frac{\partial\Psi_T}{\partial r_\star}\right)^2 - \frac{1}{2}V_T(r)\Psi_T^2\right), \tag{2.64}$$

where the potential is

$$V_T(r) = f\frac{L(L+D-3)+2(D-3)}{r^2} + f^2\frac{D(D-14)+32}{4r^2} + ff'\frac{D-6}{2r} - \frac{4\Lambda f}{D-2}. \tag{2.65}$$

As in the other cases, in frequency space the equation of motion for the radial degree of freedom takes the form of a Schrödinger equation

$$\frac{\mathrm{d}^2\Psi_T}{\mathrm{d}r_\star^2} + \Big(\omega^2 - V_T(r)\Big)\Psi_T = 0, \tag{2.66}$$

with the potential given in (2.65).

Summary. To summarize, the spherical symmetry of the background allowed us to decouple the equations for the propagating modes. We were able to separate variables, factoring out the radial profile of the fields, times an angular part which can be decomposed in full generality in terms of scalar, vector and tensor harmonics. As a result, the problem of perturbations around non-rotating black holes in general relativity has been reduced to ordinary differential equations. The equations can in turn be cast in full generality in the Schrödinger form

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r_\star^2} + \Big(\omega^2 - V(r)\Big)\Psi = 0, \tag{2.67}$$

where the potential depends on r and the quantum number L of the spherical harmonics, and schematically takes the form in figure 2.

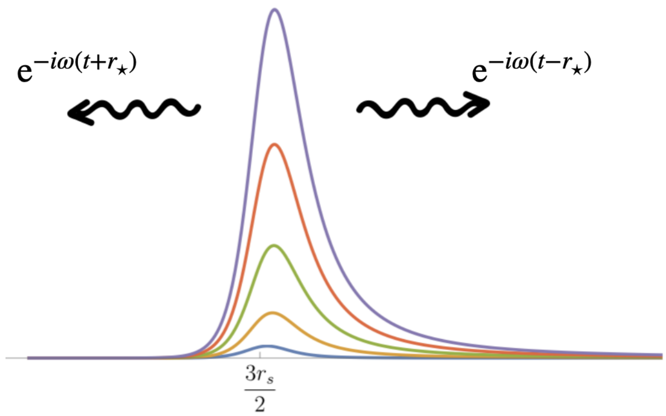

Figure 2: Potential for various values of L in $D = 4$. As a consequence of the spherical symmetry, the potential is independent of the magnetic quantum number M.

2.2 Perturbations of Kerr black holes

We will now extend the previous discussion to rotating black holes in general relativity. As we shall see, the problem of studying the dynamics of the perturbations of rotating black holes is more involved. In order to determine the equations that govern the perturbations of the Kerr metric, the most natural approach would seem to be as the one before; i.e., one would first like to expand the Kerr metric in fluctuations, plug the perturbed metric into the Einstein–Hilbert action and possibly fix a convenient gauge, and finally try to simplify the equations as much as possible to obtain a single master equation for the metric perturbation, analogous to the Regge–Wheeler and Zerilli equations. Unfortunately, the Kerr metric does not straightforwardly allow this, and in fact such a program has never been carried out in this form. In $D = 4$, an alternative route to study black hole perturbations is possible with the Newman–Penrose formalism: instead of considering perturbations of the metric, it turns out to be much more fruitful to study perturbations of the curvature tensor. This formalism makes use of the components of the Weyl tensor, which are projected along a null tetrad. Since the problem is thus significantly more involved we will focus below only on Kerr black holes in $D = 4$. Another reason for choosing to discuss only four-dimensional rotating black holes is because the landscape of rotating solutions in higher dimensions is much richer than in $D = 4$. There is no uniqueness theorem and black holes solutions with nontrivial horizon topologies are possible. Therefore, there is no general procedure that systematically applies to all cases, as instead we were able to do for the Schwarzschild–Tangherlini solution.

The Kerr line element in Boyer–Lindquist coordinates is:

$$\begin{aligned}\mathrm{d}s^2 &= -\frac{\Delta}{\rho^2}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi\right)^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(a\,\mathrm{d}t - (r^2+a^2)\mathrm{d}\varphi\right)^2 \\ &= -\frac{\rho^2 - r_s r}{\rho^2}\mathrm{d}t^2 - \frac{2ar_s r\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\mathrm{d}\varphi^2,\end{aligned} \tag{2.68}$$

where we have defined the quantities

$$\rho^2 \equiv r^2 + a^2 \cos^2\theta\,, \qquad \Delta \equiv r^2 - r r_s + a^2\,, \tag{2.69}$$

where also $\Delta = (r - r_+)(r - r_-)$ with $r_\pm \equiv r_s/2 \pm \sqrt{(r_s/2)^2 - a^2}$ being the locations of the inner and outer horizons. It is useful to note the relations: $r_+ r_- = a^2$, $a^2 + r_+^2 = r_s r_+$ and $g_{t\varphi}{}^2 - g_{tt} g_{\varphi\varphi} = \Delta \sin^2\theta$.

Null tetrads. Given a space-time with metric $g_{\mu\nu}$, a Newman–Penrose null tetrad is a set of four linearly independent four-vectors, which we will denote with

$$z_a^\mu = (l^\mu, q^\mu, m^\mu, \bar{m}^\mu), \tag{2.70}$$

where a labels the four-vectors. The four-vectors l^μ and q^μ are real, while m^μ and $\bar{m}^\mu$ are complex (one being the complex conjugate of the other). These four-vectors are chosen to be null with respect to the metric $g_{\mu\nu}$ i.e.,

$$g_{\mu\nu} l^\mu l^\nu = g_{\mu\nu} q^\mu q^\nu = g_{\mu\nu} m^\mu m^\nu = g_{\mu\nu} \bar{m}^\mu \bar{m}^\nu = 0\,. \tag{2.71}$$

In addition, we can choose them in such a way that

$$g_{\mu\nu} l^\mu q^\nu = -1, \qquad g_{\mu\nu} m^\mu \bar{m}^\nu = 1\,, \tag{2.72}$$

while all the other products are zero. From these relations and the fact that z_a^μ form a basis in $D = 4$, it follows that the (inverse) metric can be written as

$$g^{\mu\nu} = m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu - l^\mu q^\nu - l^\nu q^\mu. \tag{2.73}$$

As an example, it is straightforward to find a tetrad in flat spacetime; we can simply take

$$l^\mu = (1, 0, 0, 1), \tag{2.74}$$

$$q^\mu = \frac{1}{2}(1, 0, 0, -1), \tag{2.75}$$

$$m^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0), \tag{2.76}$$

$$\bar{m}^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0), \tag{2.77}$$

which can be used to express the Minkowski metric $\eta_{\mu\nu}$ as

$$\eta^{\mu\nu} = m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu - l^\mu q^\nu - l^\nu q^\mu. \tag{2.78}$$

In the Kerr metric a possible choice of null tetrad is given, in Boyer–Lindquist coordinates, by the Kinnersley tetrad:

$$l^\mu = \frac{1}{\Delta}(r^2 + a^2, \Delta, 0, a), \tag{2.79a}$$

$$q^\mu = \frac{1}{2\rho^2}(r^2 + a^2, -\Delta, 0, a), \tag{2.79b}$$

$$m^\mu = \frac{1}{\sqrt{2}} \frac{1}{r + ia\cos\theta}(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}). \tag{2.79c}$$

Again, these four-vectors satisfy orthogonality conditions and can be used to write the Kerr metric as in (2.73).

Newman–Penrose formalism. The Newman–Penrose formalism consists in projecting the Weyl tensor onto the null tetrad. The Weyl tensor in D dimensions is defined as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{D-2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) + \frac{1}{(D-1)(D-2)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)R. \tag{2.80}$$

Recall that the Weyl tensor has the same symmetries as the Riemann tensor: it is antisymmetric with respect to the exchange of the first and the second pair of indices,

$$C_{\mu\nu\rho\sigma} = -C_{\nu\mu\rho\sigma}, \qquad C_{\mu\nu\rho\sigma} = -C_{\mu\nu\sigma\rho}, \tag{2.81}$$

and it is symmetric under the exchange of the first and second pair,

$$C_{\mu\nu\rho\sigma} = C_{\rho\sigma\mu\nu}, \tag{2.82}$$

and satisfies cyclicity with respect to the last three indices,

$$C_{\mu\nu\rho\sigma} + C_{\mu\rho\sigma\nu} + C_{\mu\sigma\nu\rho} = 0. \tag{2.83}$$

In particular, the Weyl tensor vanishes when any pair of its indices is contracted, i.e. it is the trace-free part of the Riemann tensor.

Projecting the $D = 4$ Weyl tensor onto the null tetrad we can define the Weyl scalars [20]

$$\Psi_0 = C_{\mu\nu\rho\sigma}l^\mu m^\nu l^\rho m^\sigma, \tag{2.84a}$$
$$\Psi_1 = C_{\mu\nu\rho\sigma}l^\mu q^\nu l^\rho m^\sigma, \tag{2.84b}$$
$$\Psi_2 = \frac{1}{2}C_{\mu\nu\rho\sigma}l^\mu q^\nu\left(l^\rho q^\sigma + m^\rho \bar{m}^\sigma\right), \tag{2.84c}$$
$$\Psi_3 = C_{\mu\nu\rho\sigma}q^\mu l^\nu q^\rho \bar{m}^\sigma, \tag{2.84d}$$
$$\Psi_4 = C_{\mu\nu\rho\sigma}q^\mu \bar{m}^\nu q^\rho \bar{m}^\sigma. \tag{2.84e}$$

Similarly, one can project the Ricci tensor and define

$$\Phi_{00} = -\frac{1}{2}R_{\mu\nu}l^\mu l^\nu, \tag{2.85a}$$
$$\Phi_{01} = -\frac{1}{2}R_{\mu\nu}l^\mu m^\nu, \tag{2.85b}$$
$$\Phi_{02} = -\frac{1}{2}R_{\mu\nu}m^\mu m^\nu, \tag{2.85c}$$
$$\Phi_{11} = -\frac{1}{4}R_{\mu\nu}\left(l^\mu q^\nu + m^\mu \bar{m}^\nu\right), \tag{2.85d}$$
$$\Phi_{12} = -\frac{1}{2}R_{\mu\nu}q^\mu m^\nu, \tag{2.85e}$$
$$\Phi_{22} = -\frac{1}{2}R_{\mu\nu}q^\mu q^\nu. \tag{2.85f}$$

In vacuum $R_{\mu\nu} = 0 = R$, so (2.85) vanish for gravitational waves at large distances from the source. In general relativity, the two propagating degrees of freedom in a gravitational wave are described by the single complex quantity Ψ_4 or, equivalently, Ψ_0 which is proportional to Ψ_4^* in the radiation zone. In fact, recall that, for a gravitational wave on flat space propagating along the z axis,[8]

$$R_{0i0j} = -\frac{1}{2}\ddot{h}_{ij}^{\rm TT}, \tag{2.87}$$

with

$$h_{11} = -h_{22} \equiv h_+, \qquad h_{12} = h_{21} \equiv h_\times, \tag{2.88}$$

and the only non-vanishing Weyl scalars are

$$\Psi_0 = -\frac{1}{2}\left(\ddot{h}_+ + i\ddot{h}_\times\right), \tag{2.89a}$$

$$\Psi_4 = -\frac{1}{8}\left(\ddot{h}_+ - i\ddot{h}_\times\right), \tag{2.89b}$$

which are related by complex conjugation, $\Psi_0 = 4\Psi_4^*$.

Teukolsky equation. The Newman-Penrose formalism has been proven useful to obtain the generalization of the Regge–Wheeler and Zerilli equations for Kerr black holes [21, 22]. The Einstein equations for a perturbed Kerr metric are quite involved [20, 21]. The strategy is to introduce the tetrad $\{l^\mu, q^\mu, m^\mu, \bar{m}^\mu\}$ and decompose it in background plus perturbations as follows:

$$l^\mu \equiv l^\mu_A + l^\mu_B, \qquad q^\mu \equiv q^\mu_A + q^\mu_B, \qquad m^\mu \equiv m^\mu_A + m^\mu_B, \qquad \bar{m}^\mu \equiv \bar{m}^\mu_A + \bar{m}^\mu_B, \tag{2.90}$$

where the subscript A denotes the background value of the tetrad vectors, while the subscript B represents the perturbation. As unperturbed tetrad it is convenient to choose the Kinnersley tetrad (2.79), in such a way that[9]

$$\Psi_0^A = \Psi_1^A = \Psi_3^A = \Psi_4^A = 0. \tag{2.91}$$

The fact that $\Psi^A = 0$ implies that Ψ^B are invariant under linearized coordinate transformations. In fact, since Ψ are scalar quantities, they transform under a gauge transformation $x^\mu \to x'^\mu = x^\mu + \xi^\mu$ as $\Psi(x) \to \Psi'(x') = \Psi(x)$, i.e., to linear order in ξ,

$$\Psi'^B(x) = \Psi^B(x) - \xi^\mu \partial_\mu \Psi^A(x), \tag{2.92}$$

which means that Ψ^B does not transform if its background vanishes, $\Psi^A = 0$.

After a series of nontrivial manipulations, Teukolsky found that the perturbation equations for Ψ_0 and Ψ_4 decouple, and can be written (in vacuum) in a unified form as [21]

$$\begin{aligned}\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\partial_t^2\psi + \frac{2ar_sr}{\Delta}\partial_t\partial_\varphi\psi + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\partial_\varphi^2\psi \\ - \Delta^{-s}\partial_r\left(\Delta^{s+1}\partial_r\psi\right) - \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\psi\right) - 2s\left[\frac{a(2r-r_s)}{2\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right]\partial_\varphi\psi \\ - 2s\left[\frac{r_s(r^2-a^2)}{2\Delta} - r - ia\cos\theta\right]\partial_t\psi + (s^2\cot^2\theta - s)\psi = 0,\end{aligned} \tag{2.93}$$

[8] The transverse-traceless (TT) gauge is defined by

$$h^{0\mu} = 0, \qquad h^a{}_a = 0, \qquad \partial^a h_{ab} = 0, \tag{2.86}$$

where a, b are spatial indices in $D = 4$ Minkowski space.

[9] The derivation actually holds more in general and applies to any Type D vacuum background metric [21].

where s is the spin weight, and $\psi = \Psi_0$ for $s = +2$, while $\psi = (r - ia\cos\theta)^4 \Psi_4$ for $s = -2$. Eq. (2.93) is known as the Teukolsky equation.

Quite remarkably, the Teukolsky equation (2.93) holds beyond spin 2: the perturbation equations for massless spin-0, spin-1/2 and spin-1 fields on Kerr spacetime can all be cast in the same form (2.93). For instance, in the case of electromagnetism, one shall identify $\psi \mapsto F_{\mu\nu}\bar{m}^\mu q^\nu$ for $s = +1$ and $\psi \mapsto (r - ia\cos\theta)^2 F_{\mu\nu}\bar{m}^\mu q^\nu$ for $s = -1$, while in the scalar case $s = 0$ and ψ becomes the Klein–Gordon field. For spinors instead $s = \pm 1/2$—see [21] for a precise relation between ψ and the spinor field.

Not surprisingly, separation of variables in (2.93) does not work with spherical harmonics. Although one can formally use spherical harmonics to parametrize the most general perturbation around Kerr, the linearized dynamics of different harmonics no longer decouples for nonzero black hole spin. Nevertheless, separation of variables is possible by using spheroidal harmonics. We shall decompose ψ as

$$\psi^{(s)}(t, r, \theta, \varphi) = \int \frac{\mathrm{d}\omega}{2\pi} \sum_{\ell,m} R^{(s)}_{\ell m}(r) S^{(s)}_{\ell m}(\theta)\, \mathrm{e}^{-i\omega t + im\varphi}, \tag{2.94}$$

where $S^{(s)}_{\ell m}(\theta)$ are spin-weighted spheroidal harmonics, satisfying the differential equation

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta S^{(s)}_{\ell m}\right) + \left[a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2sa\omega\cos\theta - 2sm\frac{\cos\theta}{\sin^2\theta} - (s^2\cot^2\theta - s) + A^{(s)}_\ell(\omega)\right] S^{(s)}_{\ell m} = 0 \tag{2.95}$$

where $A^{(s)}_\ell(\omega)$ are the separation constants, which depend on the frequency and can be determined by requiring that $S^{(s)}_{\ell m}(\theta)$ is regular on the interval $\theta \in [0, \pi]$. Note that, for $s = 0$, $S^{(s)}_{\ell m}(\theta)$ reduce to the spheroidal functions, while at $a\omega = 0$ they coincide with the spin-weighted spherical harmonic $Y^{(s)}_{\ell m}(\theta)$. Note that, for generic values of the frequency ω, the separation constants $A^{(s)}_\ell(\omega)$ do not admit in general a closed-form expression, while they have to be solved for numerically (see, e.g., [23]).

Using the decomposition (2.94) where the spin-weighted spheroidal harmonics satisfy (2.95), one finds the following equation for the radial profile of ψ [21]:

$$\Delta^{-s}\partial_r\left(\Delta^{s+1}\partial_r R^{(s)}_{\ell m}\right) + \left[\frac{K^2(\omega) - is(2r - r_s)K(\omega)}{\Delta} + 4is\omega r - \lambda^{(s)}_{\ell m}(\omega)\right] R^{(s)}_{\ell m} = 0, \tag{2.96}$$

where

$$K(\omega) \equiv (r^2 + a^2)\omega - am, \tag{2.97}$$

$$\lambda^{(s)}_{\ell m}(\omega) \equiv A^{(s)}_{\ell m}(\omega) + a^2\omega^2 - 2ma\omega. \tag{2.98}$$

Some comments are in order here. First, note that, as opposed to the Regge–Wheeler and Zerilli equations, the Teukolsky equation (2.96) depends explicitly on the magnetic quantum number m. This is not surprising, as the background solution is no longer spherically symmetric: the effect of the rotation is to break the degeneracy in m, resulting in an m-dependent potential. Second, note that the potential is imaginary. In particular, in the limit $a = 0$ it does not immediately recover the Regge–Wheeler and Zerilli potentials for $s = \pm 2$: in the zero-spin limit, the Teukolsky equation reduces in fact to the Bardeen–Press equation [24]—a precursor of the more general Teukolsky equation for Kerr perturbations. In turn, the

Bardeen–Press equation can be mapped onto the Regge–Wheeler and Zerilli equations via a generalized Darboux transformation [25] (see also section 3.3 below). In addition, note that the Teukolsky equation, similarly to the Bardeen–Press equation, has a long-ranged potential. Let's introduce the tortoise coordinate

$$\frac{\mathrm{d}r_\star}{\mathrm{d}r} = \frac{r^2 + a^2}{\Delta}, \tag{2.99}$$

and let's redefine the radial field $R^{(-2)}$ as

$$R^{(-2)} \equiv \frac{\Delta}{(r^2 + a^2)^{1/2}} \tilde{R}. \tag{2.100}$$

The Teukolsky equation for $s = -2$ can then be rewritten in the canonical form

$$\frac{\mathrm{d}^2}{\mathrm{d}r_\star^2} \tilde{R} + U_T \tilde{R} = 0. \tag{2.101}$$

It is straightforward to check that that, in the large-r limit, the potential U_T scales as

$$U_T(r \to \infty) = \omega^2 - \frac{4i\omega}{r} + O(r^{-2}). \tag{2.102}$$

This should be contrasted with the Regge–Wheeler and Zerilli potentials which vanish as $1/r^2$ at large distances. The $1/r$ scaling of the potential (2.102) is not particularly nice because it makes it harder to accurately compute solutions to the separated radial equation in numerical implementations. In the 1970s, Chandrasekhar and Detweiler [26, 27] were able to show, via a generalized Darboux transformation, that it is possible to recast the Teukolsky equation in a form such that the potential has a number of attractive properties [25, 28]: in particular, it is real when ω is real, and it is short-ranged i.e., it goes to zero as $\sim 1/r^2$ at large r.

3 Symmetries of black hole perturbations

In this section, we focus on the dynamics of the perturbations of Schwarzschild and Kerr black holes in general relativity in $D = 4$, and discuss and series of symmetry properties.

3.1 Teukolsky–Starobinsky identities and spin ladders

One well-known example of duality for the perturbations of Kerr black holes is given by the Teukolsky–Starobinsky identities [29, 30]. The Teukolsky–Starobinsky identities are relations that connect solutions to the Teukolsky equation with spin index $-s$ to those with $+s$, and viceversa. In Chandrasekhar's notation [22], the identities are

$$\psi^{(-1)} = \Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger\Delta\psi^{(1)}, \qquad \psi^{(1)} = \mathcal{D}_0\mathcal{D}_0\psi^{(-1)}, \qquad \text{for spin 1,} \tag{3.1}$$

and

$$\psi^{(-2)} = \Delta^2\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger\Delta^2\psi^{(2)}, \qquad \psi^{(2)} = \mathcal{D}_0\mathcal{D}_0\mathcal{D}_0\mathcal{D}_0\psi^{(-2)}, \qquad \text{for spin 2,} \tag{3.2}$$

where

$$\mathcal{D}_0 \equiv \partial_r + i\frac{am - \omega(r^2 + a^2)}{\Delta}. \tag{3.3}$$

Note that these relations hold for any value of the frequency ω. They provide a relation between $\psi^{(+s)}$ and $\psi^{(-s)}$, but they in general do not allow to connect fields with different $|s|$. Something interesting happens however at zero frequency: when $\omega = 0$, these operations can be truncated, enabling us to increment s by unity and still obtain a solution to the static Teukolsky equation, e.g.,

$$\psi^{(0)} = E^-\psi^{(1)}, \qquad \psi^{(2)} = E_1^+\psi^{(1)}, \qquad \text{etc.,} \tag{3.4}$$

where we introduced for convenience[10]

$$E^- \equiv \partial_r\,, \qquad\qquad E_s^+ \equiv \Delta\partial_r + s(r_+ + r_- - 2r) + 2iam\,, \tag{3.5}$$

which can thus be interpreted as spin ladder operators [31].

3.2 Chandrasekhar's duality in $D = 4$

In section 2.1, we derived the equations for the perturbations of Schwarzschild black holes in generic D-dimensions (with $D \geq 4$). Interestingly, the $D = 4$ case turns out to be special because, as opposed to higher dimensions ($D > 4$), the equations for the perturbations display a series of nontrivial symmetries which constrain the dynamics and the form of the physical solutions. One notable example is the Chandrasekhar duality in $D = 4$ [22, 32], which relates sectors of opposite parity of massless spin-s perturbations in a black hole background. In particular, it provides a mapping between the Regge–Wheeler and Zerilli potentials for gravitational perturbations.[11]

[10]In this notation, the Teukolsky–Starobinsky identities (3.1) and (3.2), in the $\omega = 0$ limit, read $\psi^{(-1)} = E^-E^-\psi^{(1)}$, $\psi^{(1)} = E_0^+E_{-1}^+\psi^{(-1)}$ for spin 1, and $\psi^{(-2)} = E^-E^-E^-E^-\psi^{(2)}$, $\psi^{(2)} = E_1^+E_0^+E_{-1}^+E_{-2}^+\psi^{(-2)}$ for spin 2.

[11]The Chandrasekhar duality is realized "trivially" at the level of the Teukolsky equation. See instead Ref. [11, 33] for a discussion about partially-massless spin-2 fields on Schwarzschild-(A)dS spacetimes.

In four dimensions, the action for the physical degrees of freedom derived in section 2.1 simplifies dramatically. The Regge–Wheeler and Zerilli variables still decouple (now as a consequence of parity) and their combined action can be written as[12]

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}r_\star \left(\dot{\Psi}_{\mathrm{RW}}^2 - \left(\frac{\partial \Psi_{\mathrm{RW}}}{\partial r_\star}\right)^2 - V_{\mathrm{RW}}(r)\Psi_{\mathrm{RW}}^2 + \dot{\Psi}_{\mathrm{Z}}^2 - \left(\frac{\partial \Psi_{\mathrm{Z}}}{\partial r_\star}\right)^2 - V_{\mathrm{Z}}(r)\Psi_{\mathrm{Z}}^2 \right). \tag{3.6}$$

The Regge–Wheeler and Zerilli potentials take the simplified form

$$V_{\mathrm{RW}}(r) = f(r)\left(\frac{L(L+1)}{r^2} - \frac{3r_s}{r^3}\right), \tag{3.7}$$

$$V_{\mathrm{Z}}(r) = f(r)\left(\frac{r_s}{r^3} + \frac{2\lambda}{3r^2} + \frac{8\lambda^2(2\lambda+3) - 18\Lambda r_s^2}{3(2\lambda r + 3r_s)^2}\right), \tag{3.8}$$

where the function $f(r)$ is given by

$$f(r) = 1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}, \tag{3.9}$$

and we define

$$\lambda \equiv \frac{1}{2}(L-1)(L+2). \tag{3.10}$$

The action (3.6) possesses a duality symmetry, which follows from Chandrasekhar's observation that both the Regge–Wheeler and Zerilli potentials can be derived from a single *superpotential* [34],

$$W(r) = \frac{3r_s(r_s - r)}{r^2(3r_s + 2\lambda r)} - \frac{2\lambda(\lambda+1)}{3r_s} + \frac{\Lambda r_s\, r}{3r_s + 2\lambda r}, \tag{3.11}$$

in the sense that

$$V_{\mathrm{RW}} = W^2 + f(r)\frac{\mathrm{d}W}{\mathrm{d}r} + \kappa, \tag{3.12}$$

$$V_{\mathrm{Z}} = W^2 - f(r)\frac{\mathrm{d}W}{\mathrm{d}r} + \kappa, \tag{3.13}$$

where we have defined the constant

$$\kappa \equiv -\frac{4\lambda^2(\lambda+1)^2}{9r_s^2}. \tag{3.14}$$

This is a manifestation of the fact that the Regge–Wheeler and Zerilli potentials are partner potentials in the sense of supersymmetric quantum mechanics [35]. Rewriting the Regge–Wheeler and Zerilli potentials in terms of W, it is in fact straightforward to check that the action is invariant under the duality transformation

$$\delta\Psi_{\mathrm{Z}} = \left(\frac{\partial}{\partial r_\star} - W(r)\right)\Psi_{\mathrm{RW}}, \tag{3.15a}$$

$$\delta\Psi_{\mathrm{RW}} = \left(\frac{\partial}{\partial r_\star} + W(r)\right)\Psi_{\mathrm{Z}}, \tag{3.15b}$$

which is a true off-shell symmetry, much as electric-magnetic duality is for spin-1 fields. Incidentally, because the Regge–Wheeler and Zerilli equations are linear, this implies that the right hand sides of eqs. (3.15)

[12]Recall that in $D = 4$ the h_T perturbation in (2.6) does not propagate.

are solutions to the Zerilli and Regge–Wheeler equations respectively. The symmetry is continuous, and therefore it also gives rise to a conserved Noether current:

$$J^t = -\Psi'_{\rm Z}\dot{\Psi}_{\rm RW} - \dot{\Psi}_{\rm Z}\Psi'_{\rm RW} + W\left(\Psi_{\rm RW}\dot{\Psi}_{\rm Z} - \Psi_{\rm Z}\dot{\Psi}_{\rm RW}\right), \tag{3.16}$$

$$J^{r_\star} = \dot{\Psi}_{\rm Z}\dot{\Psi}_{\rm RW} + \Psi'_{\rm Z}\Psi'_{\rm RW} + W\left(\Psi_{\rm Z}\Psi'_{\rm RW} - \Psi_{\rm RW}\Psi'_{\rm Z}\right) - \left(W^2+\beta\right)\Psi_{\rm Z}\Psi_{\rm RW}, \tag{3.17}$$

which obeys the conservation law $\partial_t J^t + \partial_{r_\star} J^{r_\star} = 0$.

The Chandrasekhar's duality has a number of interesting consequences. The main one is the fact that it is ultimately responsible for the isospectrality of the even and odd sectors in $D = 4$ for Schwarzschild-dS black holes: as we will see below, this means that even and odd sectors have same set of quasi-normal modes (QNMs) when $\Lambda \geq 0$ [34]. In addition to isospectrality, the conservation of the current also is responsible for even and odd parity tidal Love numbers being equal in $D = 4$ (see section 6). Importantly, this symmetry does not rely on auxiliary variables having been integrated out: it is possible to uplift the Chandrasekhar duality to an off-shell symmetry of the Einstein–Hilbert action linearized around Schwarzschild [36].

An alternative perspective on the relation between the Regge–Wheeler and Zerilli potentials is that the Chandrasekhar symmetry is an example of a Darboux transformation between differential equations [25], as we will now review. From this point of view, the distinguishing feature of $D = 4$ is that a transformation can be found that preserves the boundary conditions of interest in physical situations [37]. This is also what goes wrong with AdS asymptotics: there the Chandrasekhar transformation does not preserve such boundary conditions, so the two sectors are not isospectral even in $D = 4$ [34].

3.3 Darboux transformations

Let us start considering two distinct sectors, each one containing a single degree of freedom, whose linearized dynamics is described by a one-dimensional Schrödinger-like equation of the form

$$\frac{\mathrm{d}^2\Psi_\pm}{\mathrm{d}r_\star^2} + V_\pm\Psi_\pm = 0\,. \tag{3.18}$$

The $\pm$ symbol is used to denote the two sectors, V_+ and V_- are the two potentials which will in general depend on the frequency ω, while $r_\star \in (-\infty, +\infty)$ is the variable that in general corresponds to the radial tortoise coordinate. For the moment, we shall keep $V_\pm$ generic.

Let us consider the most general linear transformation relating the on-shell fields Ψ_+ and Ψ_-. Given that the equations of motion are second order, we can write it in general as

$$\Psi_+ = \beta(r_\star)\partial_{r_\star}\Psi_- + F(r_\star)\Psi_-\,, \tag{3.19}$$

which belongs to the class of the so-called (generalized) Darboux transformations discussed in [25] and originally introduced by G. Darboux in [38]. In (3.19), β and F are functions of $r_\star$ and they are assumed to be regular as $r_\star \to \pm\infty$.

Plugging (3.19) into the equation for Ψ_+ and using the Ψ_-'s equation of motion, one can derive the following constraints on β and F [25]:

$$2\partial_{r_\star}F + \partial^2_{r_\star}\beta + \beta(V_+ - V_-) = 0\,, \tag{3.20a}$$
$$\partial^2_{r_\star}F - \beta^{-1}\partial_{r_\star}(\beta^2 V_-) + F(V_+ - V_-) = 0\,. \tag{3.20b}$$

Solving for F, after simple manipulations one finds the following integro-differential equation for β,

$$\frac{\partial^3_{r_\star}\beta + 2(V_+ + V_-)\partial_{r_\star}\beta + \beta\partial_{r_\star}(V_+ + V_-)}{V_+ - V_-} = \int \mathrm{d}r_\star\,\beta(V_- - V_+)\,. \tag{3.21}$$

Thus, looking in general for a duality between even and odd sector amounts to solving the integro-differential equation (3.21) for β—or, equivalently, the fourth-order differential equation for β obtained after taking the derivative of (3.21).

Alternatively, Eqs. (3.20) can be combined in a Riccati equation,

$$F^2 + F\partial_{r_\star}\beta - \beta\partial_{r_\star}F + \beta^2 V_- = \text{constant}\,, \tag{3.22}$$

where the right-hand side denotes an integration constant that results from removing an overall derivative in $r_\star$. Note that the quantity on the left-hand side of (3.22) is precisely the proportionality factor between the Wronskians $\mathcal{W}_\pm$ associated with the equations (3.18). Indeed, denoting with $\Psi^{(1,2)}_\pm$ any two linearly independent solutions in each sector, then, using (3.19), one finds

$$\mathcal{W}_+ \equiv \Psi^{(1)}_+\partial_{r_\star}\Psi^{(2)}_+ - \Psi^{(2)}_+\partial_{r_\star}\Psi^{(1)}_+ = \left(F^2 + F\partial_{r_\star}\beta - \beta\partial_{r_\star}F + \beta^2 V_-\right)\mathcal{W}_-\,. \tag{3.23}$$

The Riccati equation (3.22) implies that $\mathcal{W}_+ = \text{constant}\times\mathcal{W}_-$, which guarantees that, if the transformation preserves the boundary conditions, then the two sectors have a common set of QNMs, defined as the values of the frequency for which the Wronskians vanish [39–41].[13]

The Chandrasekhar relation of section 3.2 [22, 32, 42] for the massless spin-2 field belongs to the subclass of transformations (3.19) with $\beta \equiv 1$ [25],

$$\Psi_+ = \partial_{r_\star}\Psi_- + F(r_\star)\Psi_-\,. \tag{3.24}$$

In this case, (3.21) becomes a consistency condition for the potentials $V_\pm$:

$$\frac{\partial_{r_\star}(V_+ + V_-)}{V_+ - V_-} = \int \mathrm{d}r_\star(V_- - V_+)\,, \tag{3.25}$$

which is famously satisfied by the Regge-Wheeler and Zerilli potentials $V_{\mathrm{RW/Z}}$. The form of the Darboux transformation is then unambiguously fixed in terms of the potentials by

$$F = \frac{\partial_{r_\star}(V_+ + V_-)}{2(V_+ - V_-)}\,. \tag{3.26}$$

[13] It is worth emphasizing that the proportionality factor between the Wronskians is constant in $r_\star$, but it can, and in general will, depend on the frequency ω. The values of ω for which this constant factor vanishes are usually referred to as algebraically special modes, which we will disregard in the following. More details can be found e.g. in [34].

4 Quasi-Normal Modes

4.1 Quasi-normal modes vs. normal modes

In normal-mode analysis, one usually deals with a finite system whose evolution is described by an ordinary differential equation (ODE), or a systems of ODEs, which are solved by imposing certain boundary conditions, e.g. the vanishing of the wavefunction outside a finite region of space. The result is a discrete spectrum of real frequencies which correspond to the normal modes of the system.

Example: The standard example is a finite, one-dimensional string with fixed ends. The system is described by a self-adjoint differential operator with discrete spectrum corresponding to a complete set of normal modes. If L is the length of the string, ρ the density and T the tension, the propagation of vibrational modes in the string is described by the wave equation

$$\frac{\partial^2\psi(t,x)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\psi(t,x)}{\partial t^2} = 0, \tag{4.1}$$

where $v^2 \equiv T/\rho$. Imposing the Dirichlet boundary conditions $\psi(t,0) = \psi(t,L) = 0$ at the ends of the string, the solution to the wave equation can be expressed as

$$\psi(t,x) = \sum_n A_n\, \mathrm{e}^{-i\omega_n t}\, \psi_n(x), \qquad \psi_n(x) = \sin(\omega_n x/v) \tag{4.2}$$

with

$$\omega_n = \frac{\pi n}{L}\sqrt{\frac{T}{\rho}}, \qquad n \in \mathbb{N}, \tag{4.3}$$

where $\omega_n \in \mathbb{R}$.

Perturbations of black holes or neutron stars are different with respect to the example above on various aspects (see, e.g., [34, 39, 40, 43] for some reviews on the topic). First of all, perturbations propagate through all space and can escape to infinity. In this sense, a relativistic compact object is more similar to an open system, e.g. waves on an infinite string. However, differently from a scattering problem in the presence of a potential barrier, as we shall see, the boundary conditions in the black hole problem force the spectrum to be complex, like in dissipative systems.[14]

Let us go back to the potential in figure 2 and consider the differential equation (2.67). We are interested in solving the following boundary-value problem: perturbations satisfy outgoing conditions at $r_\star \to +\infty$ and infalling conditions at $r_\star \to -\infty$; i.e.,

$$\Psi(r_\star \to +\infty) \to \mathrm{e}^{-i\omega(t-r_\star)}, \tag{4.4}$$

$$\Psi(r_\star \to -\infty) \to \mathrm{e}^{-i\omega(t+r_\star)}\,. \tag{4.5}$$

[14] In contrast, in the Newtonian limit, a non-relativistic star made of an ideal fluid, in the absence of internal dissipation and of gravitational radiation that carries away energy, has a real spectrum of undamped oscillations.

As opposed to a scattering problem, where one is interested e.g. in computing reflected and transmitted amplitudes given an incoming wave, there are only outgoing waves here. Eq. (4.5) follows from the requirement that classically nothing can leave the horizon, while (4.4) is because we are interested in studying the perturbations of isolated objects and their emitted gravitational waves—of course, (4.4) will be modified if one is interested in studying a different problem, such as for instance (dark) matter accretion of a black hole immersed in an external environment.

4.2 Quasi-normal modes as poles of the Green's function

The QNM contribution to the response of the perturbed black hole can be understood by studying the poles of the Green's function of the system [34, 40, 44–46]. Imagine of perturbing a black hole, e.g. by simply throwing a small mass into it. In the end, the problem boils down to solving an inhomogeneous differential equation of the form

$$\left(\partial_t^2 - \partial_x^2 + V(x)\right)\Psi(t,x) = J(t,x), \tag{4.6}$$

where x generically denotes a spatial coordinate. In the QNM problem, one shall identify x with the tortoise coordinate—compare eq. (4.6) with (2.67), where we simply added a source term to the right-hand side to describe the external perturbation.

The solution to the inhomogeneous equation (4.6), which describes the evolution of the black hole in the presence of an external perturbing source, can be found by standard Green's function methods as follows:

$$\Psi(t,x) = \int \mathrm{d}t'\mathrm{d}x'\, G(t,x;t',x')J(t',x'), \tag{4.7}$$

where $G(t,x;t',x')$ satisfies

$$\left(\partial_t^2 - \partial_x^2 + V(x)\right)G(t,x;t',x') = \delta(t-t')\delta(x-x'), \tag{4.8}$$

subject to the boundary condition

$$G = 0, \qquad \text{if } t < t'. \tag{4.9}$$

The Green's function only depends on the difference $t-t'$. We will assume that the initial data has compact support—or, it is at least sufficiently localized. We can introduce the Laplace transform

$$G(s,x;x') = \int_0^\infty \mathrm{d}t\; \mathrm{e}^{-st}\, G(t,x;0,x'), \tag{4.10}$$

such that we can rewrite the eq. (4.8) as

$$\mathcal{L}_x G(s,x;x') \equiv \left(-\partial_x^2 + s^2 + V(x)\right)G(s,x;x') = \delta(x-x'). \tag{4.11}$$

The inverse is

$$G(t,x;t',x') = \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}s}{2\pi i}\; \mathrm{e}^{s(t-t')}\, G(s,x;x'), \tag{4.12}$$

where the integral runs vertically in the complex plane, i.e. parallel to the imaginary axis, with a positive real part in such a way that all the singularities of $G(s,x;x')$ lie on its left. We will ask that $G(s,x;x')$ stays bounded in $\mathrm{Re}(s) > 0$, as required by $G(t,x;t',x') = 0$ for $t < t'$.

Construction of the Green's function. We shall now give a constructive procedure to obtain the Green's function $G(t,x;t',x')$ [40, 47]. Recall that, for $x \neq x'$, $G(s,x;x')$ solves $\mathcal{L}_x G(s,x;x') = 0$. Thus, for both $x > x'$ and $x < x'$, we can express $G(t,x;t',x')$ in terms of solutions of the homogeneous equation. Take two independent solutions, say $\Psi_\pm(s,x)$,[15] such that

$$\Psi_+(s, x \to -\infty) \to \mathrm{e}^{sx}, \tag{4.13}$$

$$\Psi_-(s, x \to +\infty) \to \mathrm{e}^{-sx}\,. \tag{4.14}$$

Boundedness of $G(s,x;x')$ in the right half of the complex plane, $\mathrm{Re}(s) > 0$, requires that

$$G(s,x;x') \propto \Psi_-(s,x), \qquad \text{for } x > x', \tag{4.15}$$

$$G(s,x;x') \propto \Psi_+(s,x), \qquad \text{for } x < x'. \tag{4.16}$$

The x'-dependent coefficients can then be determined by imposing the correct junction conditions at $x = x'$. Continuity of G across $x = x'$ implies that

$$G(s,x;x') = A(x') \left[\Psi_+(s,x')\Psi_-(s,x)\Theta(x-x') + \Psi_-(s,x')\Psi_+(s,x)\Theta(x'-x)\right], \tag{4.17}$$

where Θ is the Heaviside step function. Integrating the Green's function equation in a neighborhood of x', and using the continuity of G, one then finds

$$\lim_{\varepsilon \to 0} \left[-\partial_x G(s,x;x')\big|_{x=x'+\varepsilon} + \partial_x G(s,x;x')\big|_{x=x'-\varepsilon} \right] = 1, \tag{4.18}$$

which fixes the overall $A(x')$:

$$\frac{1}{A(x')} = -\Psi_+(s,x')\partial_{x'}\Psi_-(s,x') + \Psi_-(s,x')\partial_{x'}\Psi_+(s,x') \equiv \mathcal{W}[\Psi_+,\Psi_-](s), \tag{4.19}$$

which coincides with the Wronskian of the homogeneous Schrödinger-like equation. All in all,

$$G(s,x;x') = \frac{1}{\mathcal{W}} \left[\Psi_+(s,x')\Psi_-(s,x)\Theta(x-x') + \Psi_-(s,x')\Psi_+(s,x)\Theta(x'-x)\right]. \tag{4.20}$$

Note that the Wronskian is conserved, i.e. $\partial_x \mathcal{W} = 0$. The Wronskian depends on s but not on the spatial coordinate x.

We are now interested in understanding the analytic structure of $G(s,x;x')$. Let's start by focusing on the roots of $\mathcal{W}(s)$, which correspond to the poles of the Green's function. Note that $\mathcal{W}(s) = 0$ happens whenever $\Psi_- \propto \Psi_+$. This means that the values of s where $\mathcal{W}(s) = 0$ are such that the solution for Ψ simultaneously satisfies outgoing conditions at both boundaries: these are precisely the quasi-normal modes, according to the definition above.

Note that, for real potentials, the QNMs always come in complex conjugate pairs: if $\Psi(s_{\mathrm{QNM}},x)$ is a QNM solution, then $\Psi^*(s^*_{\mathrm{QNM}},x)$ solves the same equation.

[15] Although we use the same notation, $\Psi_\pm$ here should not be confused with the $\Psi_\pm$ of section 3.3.

Putting everything together:

$$\Psi(t,x) = \int \mathrm{d}t'\mathrm{d}x' \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}s}{2\pi i}\, \mathrm{e}^{s(t-t')}\, G(s,x;x')J(t',x') = \int \mathrm{d}x' \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}s}{2\pi i}\, \mathrm{e}^{st}\, G(s,x;x')J(s,x'). \quad (4.21)$$

If the analytic structure of the Green's function consisted in just simple poles, given by the zeros of $\mathcal{W}(s)$, for a well-behaved $G(s,x;x')$ in the limit $|s| \to \infty$ with $\mathrm{Re}(s) < 0$, we would compute the integral in (4.21) by closing the contour on the left of the complex plane and would find

$$\Psi(t,x) \stackrel{?}{=} \sum_q \mathrm{e}^{s_q t}\, \mathrm{Res}\left(\frac{1}{\mathcal{W}(s)}, s_q\right) \left[\Psi_-(s_q,x) \int_{-\infty}^{x} \mathrm{d}x' \Psi_+(s_q,x')J(s_q,x') - \Psi_+(s_q,x) \int_{x}^{\infty} \mathrm{d}x' \Psi_-(s_q,x')J(s_q,x')\right]. \quad (4.22)$$

For instance, assuming that the initial data has compact support, with $J(s,x')$ nonzero in $x' \in [x_1,x_2]$, taking $x > x_2$ we could write

$$\Psi(t,x) \stackrel{?}{=} \sum_q c_q\, \mathrm{e}^{s_q t}\, \Psi_-(s_q,x), \qquad c_q \equiv \left(\frac{\mathrm{d}\mathcal{W}(s_q)}{\mathrm{d}s}\right)^{-1} \int_{x_1}^{x_2} \mathrm{d}x' \Psi_+(s_q,x')J(s_q,x'), \quad (4.23)$$

to be contrasted with eq. (4.2). The final result would then appear to mirror exactly the normal mode case in (4.2), where the solution is expressed as a sum of 'normal' functions which form a basis, the only difference being that here s_q are not real. In this picture, just like for normal modes, QNMs are seen as a property of the black hole or the neutron star, and describe how the system oscillates in response to an external perturbation. Despite the similarity, there are however some crucial differences. First of all, the definition of completeness cannot be directly translated to the QNM problem [39, 48]: in fact, the domain of the operator typically extends to infinity, where the QNM solutions diverge. In addition, while Ψ_+ is an analytic function throughout the complex s-plane, the solution Ψ_- has an essential singularity at $s = 0$. Furthermore, there is a branch cut extending from $s = 0$ to $-\infty$. This difference between Ψ_- and Ψ_+ can be traced back to the fact that the potential falls off exponentially in x as $x \to -\infty$, while it decays as $1/x^2$ for $x \to +\infty$. The analytic structure in the complex plane is summarized in figure 3 [40]. Finally, there is an additional contribution due to the arcs of the semi-infinite circle of the integration contour: this carries information about high-frequency signals, which are insensitive to the potential and effectively propagate in flat space [40, 49].

Despite these complications, for practical purposes, QNMs provide an accurate enough description of the ringdown, before the power-law tail takes over. As we shall see explicitly, while the tail depends on the long-range behavior of the potential, the QNM spectrum is mainly determined by the form of the potential around the maximum of the potential.

Summary: To summarize, for a system with an incomplete set of QNMs, eq. (4.23) is modified as

$$\Psi(t,x) = \sum_q c_q\, \mathrm{e}^{s_q t}\, \Psi_-(s_q,x) + (\text{other contributions}), \quad (4.24)$$

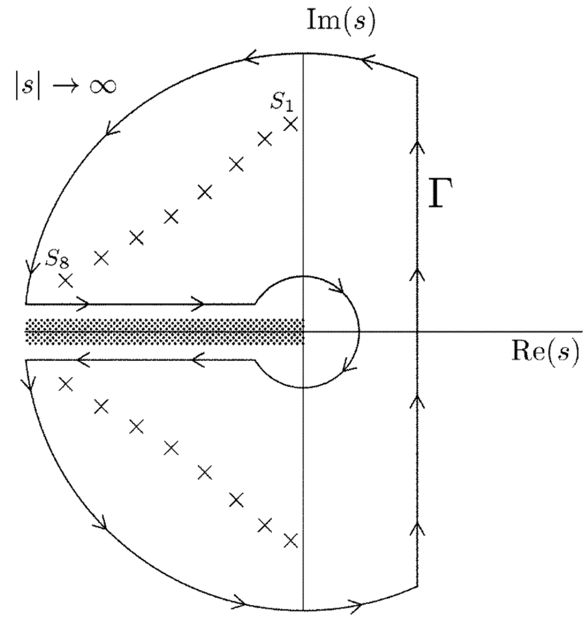

Figure 3: Picture taken from [40].

where 'other contributions' refers to the additional features in the analytic structure of the solution, other than simple poles. All in all, at the level of the time-domain Green's function, the final result can be separated into three qualitatively distinct pieces: the piece of the Green's function associated with the QNM poles where the Wronksian vanishes; the contribution from the branch cut (which tends to be subdominant compared to the QNM poles at intermediate times); and the flat piece of the Green's function from the arcs at infinity.

The QNMs are labeled in general by three numbers: the overtone n, and the quantum numbers ℓ and m.

4.3 Computing quasi-normal modes

It is instructive to consider a simple toy model that admits an exactly solvable spectrum of QNMs.

Example: Let's consider a field Ψ with dynamics given by [34]

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r_\star^2} + \left[\omega^2 - \frac{V_0}{\cosh^2\left(\alpha(r_\star - \bar{r}_\star)\right)}\right]\Psi = 0, \tag{4.25}$$

where α, V_0 and $\bar{r}_\star$ constant parameters such that

$$\frac{\mathrm{d}V}{\mathrm{d}r_\star}(r_\star = \bar{r}_\star) = 0, \qquad V_0 = V(\bar{r}_\star), \qquad \alpha^2 = -\frac{1}{2V_0}\frac{\mathrm{d}^2V}{\mathrm{d}r_\star^2}(r_\star = \bar{r}_\star). \tag{4.26}$$

Defining $\xi \equiv [1 + \mathrm{e}^{-2\alpha(r_\star - \bar{r}_\star)}]^{-1}$, we shall rewrite

$$\xi^2(1-\xi)^2 \frac{\mathrm{d}^2\Psi}{\mathrm{d}\xi^2} - \xi(1-\xi)(2\xi - 1)\frac{\mathrm{d}\Psi}{\mathrm{d}\xi} + \left[\frac{\omega^2}{4\alpha^2} - \frac{V_0}{\alpha^2}\xi(1-\xi)\right]\Psi = 0. \tag{4.27}$$

Moreover, defining

$$a \equiv \frac{1}{2\alpha}\left[\alpha + \sqrt{\alpha^2 - 4V_0} - 2i\omega\right], \qquad b \equiv \frac{1}{2\alpha}\left[\alpha - \sqrt{\alpha^2 - 4V_0} - 2i\omega\right], \qquad c \equiv 1 - \frac{i\omega}{\alpha}, \tag{4.28}$$

and $\Psi = [\xi(1-\xi)]^{-i\omega/(2\alpha)}\phi$, the equation takes the standard hypergeometric form

$$\xi(1-\xi)\phi'' + [c - (a+b+1)\xi]\phi' - ab\phi = 0. \tag{4.29}$$

The two independent solutions can be written in the form

$$\Psi = A\xi^{i\omega/(2\alpha)}(1-\xi)^{-i\omega/(2\alpha)}\,{}_2\mathsf{F}_1\,(a-c+1, b-c+1; 2-c; \xi) + B[\xi(1-\xi)]^{-i\omega/(2\alpha)}\,{}_2\mathsf{F}_1\,(a, b; c; \xi). \tag{4.30}$$

Recall that: at spatial infinity (i.e., $r_\star \to +\infty$) $\xi \to 1$ and $1-\xi \sim \mathrm{e}^{-2\alpha r_\star}$; at the horizon (i.e., $r_\star \to -\infty$) $\xi \to 0$ and $\xi \sim \mathrm{e}^{2\alpha r_\star}$. Thus, since ${}_2\mathsf{F}_1(a, b; c; 0) = 1$ and $[\xi(1-\xi)]^{-i\omega/(2\alpha)} \sim \mathrm{e}^{-i\omega r_\star}$ while $\xi^{i\omega/(2\alpha)} \sim \mathrm{e}^{i\omega r_\star}$ at the horizon, the first solution in Ψ represents an outgoing wave, while the second one is infalling. We shall thus set $A = 0$ and

$$\begin{aligned}\Psi &= B[\xi(1-\xi)]^{-i\omega/(2\alpha)}\,{}_2\mathsf{F}_1\,(a, b; c; \xi)\\ &= B[\xi(1-\xi)]^{-i\omega/(2\alpha)}\Bigg[(1-\xi)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,{}_2\mathsf{F}_1\,(c-a, c-b; c-a-b+1; 1-\xi)\\ &\qquad + \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2\mathsf{F}_1\,(a, b; a+b-c+1; 1-\xi)\Bigg]\end{aligned} \tag{4.31}$$

where we used the identities of the hypergeometric function. As $\xi \to 1$,

$$\Psi(\xi \to 1) \to B\,\mathrm{e}^{i\omega r_\star}\left[\mathrm{e}^{-2\alpha r_\star(c-a-b)}\,\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} + \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right]. \tag{4.32}$$

Requiring $\Psi(\xi \to 1) \sim \mathrm{e}^{i\omega r_\star}$ implies that $\frac{1}{\Gamma(a)} = 0$, or $\frac{1}{\Gamma(b)} = 0$, which are satisfied for

$$\omega_n = \pm\sqrt{V_0 - \frac{\alpha^2}{4}} - \frac{i\alpha}{2}(2n+1), \qquad n = 0, 1, 2, \ldots \tag{4.33}$$

In contrast with the toy example above, in physical situations, such as Schwarzschild or Kerr black holes in general relativity, the QNMs cannot be computed analytically. One has to resort to numerical or approximate methods. The most common approaches include: eikonal limit, inverted potential method [50–52], WKB approach [53], Leaver method (or continued-fraction method) [54], direct integration (or shooting method), time evolution, uniform approximation techniques [55]. See e.g. [34, 43, 56] for some reviews.

In the following, I will review the semi-analytic method introduced in [53]. However, instead of following the standard approach of Schutz and Will [53], I will adopt a slightly different, but equivalent, perspective used in [57].

The WKB approximation. Just like vibration modes in a medium originating from a perturbed object, QNMs can be thought of as waves moving around a black hole. More precisely, QNMs are interpreted as waves that originate at the light ring (the unstable circular null geodesic). This can be made more rigorous in terms of the WKB approximation [53].

It is convenient to think of the Schrödinger-like equation (2.67),

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r_\star^2} + \Big(\omega^2 - V(r)\Big)\Psi = 0, \tag{4.34}$$

in terms of an Hamiltonian formulation [57]. In order to make contact with the standard notation of Hamilton's mechanics, let's map the variable $r_\star$ to a fictitious time coordinate τ and the fields Ψ to the generalized coordinates q, and introduce the canonical conjugate momentum p. The time dependent Hamiltonian

$$H(q,p,\tau) = \frac{1}{2}p^2 + \frac{1}{2}\left(\omega^2 - V(\tau)\right)q^2, \tag{4.35}$$

reproduces (4.34), as it can be easily checked by using Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} = p, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -\left(\omega^2 - V(\tau)\right)q. \tag{4.36}$$

Hamilton's equations written in matrix form are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 + V(\tau) & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}. \tag{4.37}$$

The boundary conditions for the QNMs are

$$q \sim \begin{cases} \mathrm{e}^{+i\omega\tau} & \text{for} \quad \tau \to +\infty, \\ \mathrm{e}^{-i\omega\tau} & \text{for} \quad \tau \to -\infty. \end{cases} \tag{4.38}$$

When multiplied by the imaginary unit i, (4.37) is formally the time-dependent Schrödinger equation with a non-hermitian evolution operator. Since $|\tau\, V(\tau)| \to 0$ for $\tau \to \pm\infty$, it is natural to perform a (complex) canonical transformation that diagonalizes the time evolution in the asymptotic region. Defining

$$\xi^{\pm} \equiv \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\, q \pm i\frac{p}{\sqrt{\omega}}\right), \tag{4.39}$$

the evolution equation becomes

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix} = \left[\begin{pmatrix} +\omega & 0 \\ 0 & -\omega \end{pmatrix} + \frac{V(\tau)}{2\omega}\begin{pmatrix} +1 & +1 \\ -1 & -1 \end{pmatrix}\right]\begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix}, \tag{4.40}$$

while the boundary conditions are simply

$$\begin{cases} \xi^+ \to 0, & \xi^- \sim \mathrm{e}^{\mathrm{i}\omega\tau}, \qquad \text{for} \quad \tau \to +\infty \\ \xi^+ \sim \mathrm{e}^{-\mathrm{i}\omega\tau}, & \xi^- \to 0, \qquad\quad \text{for} \quad \tau \to -\infty. \end{cases} \tag{4.41}$$

In this basis it becomes transparent that for ω real in the asymptotic region $|\tau| \to \infty$ the evolution is unitary and governed by the asymptotic eigenvalues $\pm\omega$. The boundary conditions correspond to the requirement that the system undergoes a transition from an eigenstate with eigenvalue $+\omega$ at early times to one with eigenvalue $-\omega$ at late times. In the case of a slowly varying Hermitian evolution operator, the adiabatic theorem of quantum mechanics implies that transitions occur efficiently only when two eigenvalues become (almost) degenerate, as in the well-known Landau–Zener effect (i.e. close to a level crossing, usually lifted by perturbations in a quantum mechanical system).

The eigenvalues are:

$$\lambda_\pm = \pm\sqrt{\omega^2 - V}. \tag{4.42}$$

For a potential of the form discussed above (see, e.g., figure 2), single level crossing occurs when

$$\omega^2 = V_{\max}, \tag{4.43}$$

where $V_{\max}$ denotes the maximum of the potential V. Alternatively, this can be heuristically justified also by noticing that the QNM boundary conditions can be fulfilled by requiring the outgoing waves to have equal amplitudes both at the horizon and spatial infinity [53]: this can be accommodated only if $\omega^2 \simeq V_{\max}$ (otherwise, for $\omega^2 < V_{\max}$, there would be a region where the modes are exponentially damped).

Expanding the potential in the wave equation around the maximum:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r_\star^2} + \left[\omega^2 - V_{\max} - \frac{1}{2}V''(\bar{r}_\star)(r_\star - \bar{r}_\star)^2\right]\Psi = 0, \tag{4.44}$$

where $\bar{r}_\star$ denotes the location of the maximum i.e., $V_{\max} \equiv V(\bar{r}_\star)$. This equation admits an exact solution in terms of parabolic cylinder functions [53],

$$\Psi = AD_\nu(z) + BD_{-\nu-1}(iz), \qquad z \equiv \left[2V''(\bar{r}_\star)\right]^{1/4} \mathrm{e}^{\frac{i\pi}{4}} \left(r_\star - \bar{r}_\star\right), \tag{4.45}$$

with $\nu \equiv -i(\omega^2 - V_{\max})/\sqrt{2V''(\bar{r}_\star)} - 1/2$. Using the asymptotic expansion of the cylinder functions, and requiring outgoing boundary conditions at spatial infinity and infalling waves at the horizon, one finds the condition

$$\frac{1}{\Gamma(\nu)} = 0. \tag{4.46}$$

This implies a "Bohr–Sommerfeld quantization rule" for the QNM frequencies,

$$\frac{\omega^2 - V_{\max}}{\sqrt{2V''(\bar{r}_\star)}} = i\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \ldots \tag{4.47}$$

corresponding to the leading-order WKB approximation.

Subleading corrections can be systematically computed but taking into account higher orders in the expansion [34]. It is worth stressing that the WKB approach is not particularly efficient for precise calculations of the QNM frequencies. For high precision calculations of the frequencies one can resort instead to the Leaver method or uniform approximation techniques. The WKB approximation works better and better for large ℓ, but becomes less accurate for large overtones.

4.4 Nonlinear corrections to quasinormal modes

In the previous part of this section, we reviewed the derivation of QNMs and discussed how to compute them from the equations that describe the linearized dynamics of black hole perturbations. Since such perturbations decay exponentially, it is reasonable to expect that linear black hole perturbation theory can reasonably well model the ringdown after a time roughly of the size of the QNM decay timescale. This is in fact what ringdown analyses of gravitational-wave signals usually do. On the other hand, we know that general relativity is intrinsically nonlinear. This implies that during the initial phase of the ringdown, right after the peak of the strain, it is important to take nonlinear corrections into account. Second-order corrections in perturbation theory to the spectrum of QNMs are usually referred to as *quadratic QNMs* (QQNMs)—see, e.g., [49, 58–67] and references therein.

From a phenomenological point of view, it is worth noticing that, since the decay rate of the linear QNMs grows quickly with the overtone number n, there could be quadratic QNMs whose decay rate is slower than linear overtones. One common example of relevant QQNM is the one with harmonic numbers ($\ell = 4, |m| = 4$): this is mainly sourced by the product of two fundamental linear QNMs with ($\ell = 2, |m| = 2$), which turn out to be the dominant modes generated from the merger of nearly equal-mass binary black holes [49].

The idea is the straightforward generalization of what we discussed in section 2. Instead of expanding to linear order, we now write the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(2)}_{\mu\nu} + \ldots \tag{4.48}$$

where ε is a small bookkeeping parameter which counts the order of perturbation theory—one can think of it physically also as the small magnitude of linear modes. For simplicity, we will take here $\bar{g}_{\mu\nu}$ to be the Schwarzschild metric in $D = 4$:

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\big), \tag{4.49}$$

with $f(r) = 1 - r_s/r$. We now expand the Einstein equations up to second order in ε. We get schematically:

$$G^{(1)}_{\mu\nu}\big[h^{(1)}\big] = 0\,, \tag{4.50}$$

$$G^{(1)}_{\mu\nu}\big[h^{(2)}\big] = -G^{(2)}_{\mu\nu}\big[h^{(1)}, h^{(1)}\big]\,, \tag{4.51}$$

$$\vdots$$

where $G^{(1)}_{\mu\nu}[\cdot]$ denotes the linear variation of the Einstein tensor linear and $G^{(2)}_{\mu\nu}\big[\cdot\,,\cdot\big]$ the quadratic part in perturbations.

Conceptually, the procedure then mirrors the one followed to derive the Regge–Wheeler and Zerilli equations:

- one first fixes a convenient gauge (at first and second order) to remove the redundancy associated with diffeomorphism invariance;

- one introduces two master scalar variables $\Psi_{\pm}^{(2)}$—these are the second-order analogs of the Regge–Wheeler and Zerilli fields—which are functions of the even and odd metric perturbations, and capture the dynamics of the physical degrees of freedom;
- one solves the equation for each $\Psi_{\pm}^{(2)}$ subject to the usual QNM boundary conditions, which can later be re-expressed in terms of the metric perturbations in the physical transverse-traceless (TT) gauge, where the amplitude of the physical gravitational-wave observable at infinity can be computed.

In the end, the equations for $\Psi_{\pm}^{(2)}$ will look like

$$\frac{\mathrm{d}^2\Psi_{\pm}^{(2)}}{\mathrm{d}r_\star^2} + \left(\omega^2 - V_{\pm}\right)\Psi_{\pm}^{(2)} = S_{\pm}^{(2)} \,, \tag{4.52}$$

where the left-hand side is identical to the Regge–Wheeler and Zerilli equations, while a source term $S_{\pm}^{(2)}$ quadratic in the linear perturbations now appears on the right-hand side. As a result, one finds a set of quadratic QQNMs sourced by the product of two linear modes. The second-order frequencies are fully fixed from the combination of two first-order frequencies as [49, 66]:

$$\omega^{(2)} = \omega_{\ell_1 m_1 n_1} + \omega_{\ell_2 m_2 n_2}, \qquad \text{or} \qquad \omega^{(2)} = \omega_{\ell_1 m_1 n_1} - (\omega_{\ell_2 m_2 n_2})^*. \tag{4.53}$$

This follows because the real metric perturbations at each order can be written as the real part of the complex field:

$$h_{\mu\nu}^{(j)} = \mathrm{Re}\left[h_{\mu\nu}^{c(j)}\right] , \tag{4.54}$$

i.e., eqs. (4.50) and (4.51) can be recast as [49]

$$G_{\mu\nu}^{(1)}\left[h_c^{(1)}\right] = 0 \,, \tag{4.55}$$

$$G_{\mu\nu}^{(1)}\left[h_c^{(2)}\right] = -G_{\mu\nu}^{(2)}\left[\frac{1}{2}\left(h_c^{(1)} + h_c^{(1)*}\right), h_c^{(1)}\right] \,. \tag{4.56}$$

Note that the case of the negative sign in (4.53) can be equivalently thought of as arising from combining an ordinary linear mode and a mirror mode—a mirror mode has a frequency related to a standard mode ω by complex conjugation, i.e. $-\omega^*$ [49, 68, 69].

Note also that, since the odd and even linear QNMs are isospectral, and since they can all contribute to both odd and even sectors at second order, one expects that isospectrality will hold true also for quadratic modes [49].

The idea is then to derive the inhomogeneous solution to (4.52) (the homogeneous solution is identical to the solution in the linear problem) and compute the relative amplitude of the second-order modes. One can proceed using standard Green's function methods and the approaches mentioned in section 4.3.

Note that, despite the similarity with the linear analysis, there are some interesting differences that are worth emphasizing:

- first, unlike the case of linear QNMs, whose amplitudes are free parameters that depend on the initial conditions at the merger, the amplitude of QQNMs is entirely fixed in terms of the black hole background solution in general relativity and the linear mode amplitudes;

- as discussed in the literature [58–66], choosing a second-order master scalar that is equivalent to the first-order one generally causes a divergent behavior at infinity in the solution—although this is eventually immaterial, it suggests that better field choices may be possible which can improve the extraction of the subleading ringdown effects (see, e.g., [66] for a discussion).

Recently, evidence of second-order modes has been found in numerical simulations of merging binary systems—see e.g. [70–77] for some studies on the importance of accounting for second-order effects to accurately describe the signals obtained from numerical relativity.

As an example, following the notation of [66], let's call with $\mathcal{R}$ the relative ratio of the QNM amplitudes for the physical TT-gauge gravitational fields at large distances, i.e.,

$$\mathcal{R} \equiv \frac{\mathcal{A}^{(2)}_{\ell m \mathcal{N}}}{\mathcal{A}^{(1)}_{\ell_1 m_1 \mathcal{N}_1} \mathcal{A}^{(1)}_{\ell_2 m_2 \mathcal{N}_2}}, \qquad |\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2, \tag{4.57}$$

where $\mathcal{N} \equiv (n, \mathfrak{m})$ is composed of the overtone number n and the mirror modes $\mathfrak{m} = \pm$. Ref. [66] finds for instance that $\mathcal{R} \simeq 0.154\, \mathrm{e}^{-0.068i}$ for $(220+) \times (220+) \to (44)$, which turns out to be the most excited nonlinear mode in numerical relativity simulations, see e.g. [70–74, 76].

5 Effective field theory approach to black hole dynamics

Let us remind ourselves the sketch of the merger of two compact objects (see figure 1) [78]. Let's call r the orbit separation and $\mathcal{R}$ the typical size of the binary constituents. In the early inspiral phase, i.e. when the orbit separation is large compared to the typical size of the objects ($r \gg \mathcal{R}$), the evolution of the system is slow and admits a systematic expansion in powers of $\mathcal{R}/r$. The inspiral proceeds adiabatically following non-relativistic orbits. For black holes, which have $\mathcal{R} \sim r_s$, using virial theorem of Newtonian gravity, one finds typical relative velocities of order

$$v^2 \sim \frac{GM}{r} \sim \frac{r_s}{r} \ll 1. \tag{5.1}$$

This is the key factor behind the post-Newtonian (PN) expansion [79, 80]: the Einstein equations of the compact binary admit a systematic solution as an expansion order-by-order in the relative velocity $v \ll 1$.

The PN limit of binary dynamics has a natural formulation in the language of effective field theories [81] (see also [78, 82–87] for some reviews). In addition to the typical size $\mathcal{R} \gtrsim O(1) r_s$ of the objects in the binary, and the distance separation r between them, there is another scale in the problem: the wavelength λ of the emitted radiation, which we can roughly take to be of order of the inverse of the orbital frequency,

$$\lambda \sim \frac{r}{v}. \tag{5.2}$$

The EFT takes essentially advantage of the hierarchy between the widely separated length scales in the problem:

$$r_s \lesssim \mathcal{R} \ll r \sim \frac{r_s}{v^2} \ll \lambda \sim \frac{r}{v}. \tag{5.3}$$

Note that the three scales in the problem ($\mathcal{R}$, r and λ) are correlated i.e., their relative ratios are given by powers of the same expansion parameter v. To disentangle the physical effects entering at the three different scales $\mathcal{R}$, r and λ (i.e., at different orders in v), one can construct a 'tower' of EFTs of gravity [78, 86]: at the first stage, one constructs a one-body EFT to remove the scale $\mathcal{R}$ of the single (isolated) compact object; at the next stage, one integrates out the orbital scale of the two-body system and obtains an EFT for a composite particle; finally, one integrates out the radiation scale and obtains an effective theory of dynamical multipole moments, to describe effects that involve the radiation directly.

Before reviewing the construction of the EFT for binary systems, it is convenient to remind some aspects of effective field theories in general.

5.1 Some general considerations on effective field theories

Interesting phenomena happen at all energy scales. The existence of large separations of length scales between such phenomena is a fact in physics. This means that, in order to describe the physical properties of a system at a certain energy scale E, we do not need to know what happens at scales that are parametrically different. In other words, we can effectively "decouple" the physics at widely separated energy scales. A direct consequence of this fact is our ability to describe phenomena in terms of a finite number of parameters, which is what ultimately determines the predictive power of our description. All these aspects are formalized within the framework of Effective Field Theories (EFTs).

Operationally, the decoupling of phenomena at separated scales is obtained by *integrating out* UV physics. Concretely, if one is given a fundamental theory involving heavy fields $\Phi(x)$ and light fields $\phi(x)$, which is described by an action $S[\Phi, \phi]$, the low-energy effective action $S_{\rm EFT}[\phi]$, for the light fields only, is formally defined, within the path-integral formulation of quantum field theory, as

$$\mathrm{e}^{iS_{\rm EFT}[\phi]} \equiv \int \mathcal{D}\Phi \; \mathrm{e}^{iS[\Phi,\phi]} \; . \tag{5.4}$$

Decoupling is ultimately the statement that the resulting effective action $S_{\rm EFT}[\phi]$ is local in the light fields ϕ at energies that are much smaller than those at which the heavy fields Φ enter:

$$S_{\rm EFT} = \int \mathrm{d}^D x \sum_i c_i \frac{\mathcal{O}_i[\phi(x)]}{\Lambda^{\Delta_i - D}}, \tag{5.5}$$

where Λ is some UV energy scale, associated with the Φ fields, c_i are constant coefficients and Δ_i are the dimensions of $\mathcal{O}_i$, which are local functions of ϕ.

The definition (5.4) is concrete and predictive: one can, in particular, compute correlation functions of the light fields by performing a path integral over them, weighted by the exponential of the effective action, i.e.,

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{\displaystyle\int \mathcal{D}\phi \mathcal{D}\Phi \; \mathrm{e}^{iS[\Phi,\phi]} \; \phi(x_1) \cdots \phi(x_n)}{\displaystyle\int \mathcal{D}\phi \mathcal{D}\Phi \; \mathrm{e}^{iS[\Phi,\phi]}} = \frac{\displaystyle\int \mathcal{D}\phi \; \mathrm{e}^{iS_{\rm EFT}[\phi]} \; \phi(x_1) \cdots \phi(x_n)}{\displaystyle\int \mathcal{D}\phi \; \mathrm{e}^{iS_{\rm EFT}[\phi]}} \; . \tag{5.6}$$

At this level, to be precise, there is formally no energy expansion in (5.6): eq. (5.6) is identically true, provided that one uses the effective action for ϕ to compute correlation functions of ϕ only, but not of Φ. In other words, as long as one is only interested in correlation functions of ϕ, one can perform the path-integral over Φ once and for all, which is what eq. (5.6) does.

The point is that, from a bottom-up perspective, we often do not know what the physics of the heavy fields Φ is. That is where the power of EFTs resides: one uses the expansion (5.5), truncated at some finite order, as the starting point to describe the dynamics of ϕ. Information about the unknown UV physics is encoded in the Wilson couplings c_i, which are in general unknown: they can either be constrained experimentally, or by explicit matching with some UV theory. Consistency and predictivity of the theory rely on the fact that, up at some energy and precision, only a limited number of terms in the expansion (5.5) is required to have a sufficiently accurate description of the dynamics of ϕ.

To summarize, by taking advantage of this separation of scales, EFTs are defined by an effective Lagrangian, which is completely specified by the following building blocks (see [88–93] for some reviews on EFTs).

- *Degrees of freedom:* As first step, one identifies the low-energy degrees of freedom that will enter the effective action.

- *Symmetries:* One then specifies the symmetries, which constrain the form of the allowed operators in the effective action, and which are functions of the low-energy degrees of freedom. They can be of any type, such as global, gauged, exact, accidental, approximate, spontaneously broken, anomalous, etc.

- *Expansion parameters and power counting rules.* Once the low-energy degrees of freedom and symmetries are identified, one organizes the effective Lagrangian in powers of some expansion parameters. These take in general the form of ratios between the energy E and some Λ, the typical energy scale at which the neglected UV physics enters. The size of the higher dimensional operators that appear in the EFT is then dictated by power counting rules.

Given these ingredients, then the recipe simply consists in writing down a general Lagrangian of the form (5.5), containing all possible operators, which are built up with the low-energy degrees of freedom and are compatible with the given set of symmetries, organized in a derivative expansion. The operators $\mathcal{O}_i$ are local functions of the low-energy fields, and c_i are the IR Wilson couplings. The description (5.5) is completely general and model independent.

If a UV theory is known, given the EFT (5.5) truncated to the desired order in the derivative expansion, one can determine the IR Wilson couplings c_i by computing as many low-energy observables as needed, and by matching them to the same observables computed in the full theory. Possible non-analytic contributions in the calculation of the observables in the EFT (5.5), e.g. in the form of logarithms $\log(\Lambda/E)$, can be understood in the EFT as the renormalization group (RG) evolution of the Wilson couplings from the matching scale Λ to the IR at a scale. Such non-analytic contributions are universal, i.e. independent of the detailed microscopic physics.

5.2 Effective field theory description of compact sources

The logic summarized above naturally applies to black hole dynamics and binary systems.

In the following, we will focus on the construction of the EFT of isolated compact objects [78, 81–87, 94].[16] The compact object is treated as a dynamical point-like defects (worldline) which carries internal degrees of freedom coupled to gravity. The idea is that, far from the compact object, i.e. at distances much greater than its radius ($r \gg \mathcal{R}$), the object is seen, in first approximation, as a point particle. The EFT will be able to efficiently describe finite-size effects originating from the internal structure of the orbiting compact object, such as tidal deformations induced by external gravitational fields, or dissipation of energy across the black hole horizon. After integrating out the internal structure of the object, at distance scales larger than the typical radius $\mathcal{R}$, such effects are encoded in the Wilson coefficients of local worldline operators, suppressed by powers of $\mathcal{R}/r$. In this construction, the two-body problem can be seen as the theory of gravitons coupled to the compact object worldlines [78, 87].

According to the general considerations above, in order to write the EFT of an isolated compact object, we first have to identify the relevant low-energy degrees of freedom. For simplicity, we will assume that the internal dynamics is gapped at distance scales $\mathcal{R} \ll r$, i.e. that the only light degrees of freedom are the Goldstone modes associated with the spontaneous breaking of local Poincaré symmetries by the presence of the compact object. Thus, up to gauge redundancy, in the point-particle limit the system is described by:

- the gravitational field $g_{\mu\nu}$;

[16] See also [10, 95–99].

- the compact object's worldline $x^\mu(\tau)$, which is a function of the affine parameter τ;
- a spin degree of freedom s^i localized on the worldline.

In the following, we will denote the particle's four-velocity as

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \equiv \dot{x}^\mu. \tag{5.7}$$

(Here and in the following, we often denote derivatives with respect to the affine parameter τ with an overdot.) To remain as general as possible, we will again work in generic spacetime dimensions. Results in four dimensions can be obtained by simply setting $D = 4$ in the expressions below. We will sometimes introduce the notation

$$d \equiv D - 1, \tag{5.8}$$

for the number of spatial dimensions.

The dynamics then follows from a worldline action that couples $x^\mu(\tau)$ and the spin s^i to the gravitational field $g_{\mu\nu}$. To construct the operators in the EFT using the ingredients above, and to describe the coupling between the worldline and external fields, it is convenient to introduce an orthonormal frame [78, 85–87, 96, 100, 101]

$$e_a^\mu(\tau), \qquad\qquad a = 0, 1, 2, \ldots, d, \tag{5.9}$$

where a denotes SO$(d, 1)$ local Lorentz indices. The vielbein e_a^μ satisfies

$$g_{\mu\nu} e_a^\mu e_b^\nu = \eta_{ab}\,, \qquad \eta_{ab} e_\mu^a e_\nu^b = g_{\mu\nu}\,, \tag{5.10}$$

i.e., $e_a^\mu e_\nu^a = \delta_\nu^\mu$ and $e_a^\mu e_\mu^b = \delta_a^b$. The local frame e_a^μ is necessary to describe objects with non-zero spin. In the absence of gravity, the formalism was first introduced by Regge and Hanson [100], to treat the classical motion of relativistic spinning particles coupled to electromagnetic fields. The extension of the Regge–Hanson formalism to curved spacetime, and its applications to perturbative binary dynamics first appeared in [96].[17]

The rotation of the particle relative to fixed inertial frame is encoded in

$$\Omega^{ab} = g^{\mu\nu} e_\mu^a \frac{D}{D\tau} e_\nu^b\,, \qquad \frac{D}{D\tau} e_\mu^a = \dot{x}^\rho \nabla_\rho e_\mu^a = \dot{e}_\mu^a - \Gamma^\rho_{\sigma\mu} \dot{x}^\sigma e_\rho^a\,, \tag{5.11}$$

which corresponds to the rotation of the vielbein along the worldline's trajectory. Ω^{ab} represents the angular velocity of the particle, and is an antisymmetric matrix,

$$\Omega^{ab} = -\Omega^{ba}. \tag{5.12}$$

Following the logic of section 5.1, as a next step we shall identify the set of symmetries that will determine the form of the operators allowed in the EFT.

The compact object spontaneously breaks Poincaré symmetries, which are nonlinearly realized on the worldline degrees of freedom $\{x^\mu, e_a^\mu\}$. The symmetries that we will require in the action are those associated with gauge redundancy in the variables $\{g_{\mu\nu}, x^\mu(\tau), e_a^\mu(\tau)\}$, i.e.,

[17] A modern treatment of spinning particles from the point of view of nonlinearly realized symmetries can be found in [98].

- general coordinate invariance:

$$x^\mu \to \tilde{x}^\mu \,, \qquad g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x(\tilde{x}))\,, \qquad e^a_\mu \to \tilde{e}^a_\mu = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} e^a_\alpha\,; \tag{5.13}$$

- internal Lorentz invariance of the local frame field:

$$e^a_\mu(\tau) \to \tilde{e}^a_\mu(\tau) = \Lambda^a{}_b e^b_\mu(\tau)\,, \tag{5.14}$$

  where $\Lambda^a{}_b$ is a constant Lorentz matrix.

- reparametrization invariance (RPI) of the worldline:

$$\tau \to \tilde{\tau}(\tau)\,. \tag{5.15}$$

Because of the second point above, the definition of the tetrad is unique up to a local Lorentz transformation: we can transform the local 'a' index by a local Lorentz transformation, i.e. $\tilde{e}^a_\mu = \Lambda^a{}_b e^b_\mu$, and end up with the same metric $g_{\mu\nu}$:

$$\tilde{g}_{\mu\nu} = \eta_{ab}\tilde{e}^a_\mu \tilde{e}^b_\nu = \eta_{ab}\Lambda^a{}_c \Lambda^b{}_d e^c_\mu e^d_\nu = \eta_{cd} e^c_\mu e^d_\nu = g_{\mu\nu}, \tag{5.16}$$

where we used (5.10).

To enforce RPI, it will be convenient to define also an einbein $e(\tau)$, such that

$$\tilde{e}(\tilde{\tau})\mathrm{d}\tilde{\tau} = e(\tau)\mathrm{d}\tau. \tag{5.17}$$

Concretely, under an infinitesimal shift of the affine parameter,

$$\delta\tau = \xi, \tag{5.18}$$

the einbein, the worldline and the vielbein transform as

$$\delta e = \partial_\tau(\xi e)\,, \qquad \delta x^\mu = \xi \dot{x}^\mu\,, \qquad \delta e^a_\mu = \xi \dot{e}^a_\mu\,. \tag{5.19}$$

Worldline effective action. Having specified the degrees of freedom and the symmetries, we shall next proceed by writing down the most general RPI action.

Point-particle action of non-rotating bodies. As a warm-up, let us consider the case of a non-spinning object. From far away, the object appears, in first approximation, as a point particle. At lowest order, the action describing the dynamics of a point particle is the Nambu–Goto action on the worldline:

$$S_{\mathrm{pp}} = -m\int \mathrm{d}s = -m\int \mathrm{d}\tau\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}\,, \tag{5.20}$$

where s denotes here the proper time along the worldline, and is related to the affine parameter τ via

$$\mathrm{d}s = me\mathrm{d}\tau\,. \tag{5.21}$$

In terms of the einbein, we can in fact equivalently rewrite the Nambu–Goto action in the Polyakov form as

$$S_{\rm pp} = \frac{1}{2}\int \mathrm{d}\tau e\left(e^{-2}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}g_{\mu\nu} - m^2\right) . \tag{5.22}$$

Integrating out the einbein from (5.22), using its equation of motion,

$$e = \frac{1}{m}\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}, \tag{5.23}$$

one correctly recovers the Nambu–Goto action (5.20). Note that extremizing $S_{\rm pp}$ gives the usual geodesic motion of a test particle in a gravitational field:

$$0 = \delta\left[m\int \mathrm{d}s\right] = \delta\left[m\int\sqrt{-g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}\right] = -m\int \mathrm{d}s\left[g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}s} + \frac{1}{2}(\partial_\nu g_{\rho\sigma})\frac{\mathrm{d}x^\rho}{\mathrm{d}s}\frac{\mathrm{d}x^\sigma}{\mathrm{d}s}\right]\delta x^\nu \tag{5.24}$$

$$\Rightarrow \qquad a^\mu \equiv \frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}s}\frac{\mathrm{d}x^\sigma}{\mathrm{d}s} = \frac{\mathrm{d}x^\rho}{\mathrm{d}s}\nabla_\rho\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = 0. \tag{5.25}$$

Including the spin degree of freedom, to leading order in a derivative expansion in powers of $\mathcal{R}/r$, the point-particle action can be written as

$$S_{\rm pp} = \int \mathrm{d}\tau\left[v^\mu p_a e^a_\mu + \frac{1}{2}S^{ab}\Omega_{ab} - \frac{1}{2}e\big(p_a p^a + m^2(p,S)\big) + e\lambda_a S^{ab}p_b\right] , \tag{5.26}$$

where the momentum p^a and the spin S_{ab},

$$p_\mu \equiv \frac{\delta S_{\rm pp}}{\delta\dot{x}^\mu} , \qquad S^{ab} \equiv 2\frac{\delta S_{\rm pp}}{\delta\Omega_{ab}} , \tag{5.27}$$

are conjugate variables to the particle's velocity $\dot{x}^\mu$ and angular momentum Ω_{ab}, with

$$p_\mu = e^a_\mu p_a, \qquad S_{\mu\nu} = e^a_\mu e^b_\nu S_{ab}. \tag{5.28}$$

(In this construction we are following [101], however there is a different way of proceeding based on the Routhian formalism [102].)

In the formulation (5.26), the quantity m^2 is in principle an arbitrary function of all possible scalars constructed out of p_a, S_{ab} and $g_{\mu\nu}$. The form of this function is not predicted by the point-particle EFT, but it is rather fixed through a matching procedure to the UV theory of the extended object. Note that $m^2(p,S)$ fixes the Regge trajectory [100] of the spinning particle, i.e. the relation between the invariant mass p^2 and the spin, which follows from the variation of $S_{\rm pp}$ with respect to $e(\tau)$.

Moreover, the last term in (5.26), involving the Langrange multiplier λ_a, is necessary to enforce a supplementary constraint on S^{ab}. In particular,

$$S^{ab}p_b = 0 \qquad\qquad (\text{“spin-supplementary condition”}). \tag{5.29}$$

Recall that information about the particle's angular momentum is encoded in the angular velocity matrix Ω_{ab}, which is antisymmetric. Therefore, it has in principle $D(D-1)/2$ independent components. However, the angular momentum of classical objects belongs to the adjoint representation of the spatial rotation group $\mathrm{SO}(D-1)$, whose dimension is $(D-1)(D-2)/2$ independent components. The spin-supplementary condition is thus necessary to reduce the number of independent components of Ω_{ab} from $D(D-1)/2$ down to $(D-1)(D-2)/2$:

$$\frac{D(D-1)}{2} - (D-1) = \frac{(D-1)(D-2)}{2}, \tag{5.30}$$

as required by Poincaré symmetries for a physical spin degree of freedom.

It will be convenient later on to project onto the subspace orthogonal to the particle's motion—in the rest frame of the particle, this means projecting onto the spatial coordinates. The covariant form of such a projector is

$$P^{\mu}_{\nu} \equiv \delta^{\mu}_{\nu} + u^{\mu} u_{\nu}\,, \tag{5.31}$$

where we defined the four-velocity with respect to the proper time:

$$u^{\mu} \equiv \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s} = \frac{v^{\mu}}{\sqrt{-v^{\lambda}v_{\lambda}}}, \qquad u^{\mu}u_{\mu} = -1. \tag{5.32}$$

(To be pedantic, we introduced u^{μ} to stress that the four-velocity with respect to the proper time differs from v^{μ} by a normalization factor—this difference is however immaterial in Minkowski space where the two four-vectors coincide.) In the particle's rest frame,

$$u^{a} = e^{a}_{\mu} u^{\mu} = (1, \vec{0})\,. \tag{5.33}$$

Analogously, we can introduce the projector

$$P^{b}_{a} = \delta^{b}_{a} + u_{a} u^{b}. \tag{5.34}$$

The vielbein e^{a}_{μ} will be useful to convert spacetime indices into rest-frame indices, while we will use P^{b}_{a} to project onto spatial indices.

Bulk theory. We want eventually to couple the worldline to external fields. Therefore, we need to specify their bulk dynamics. In the case of gravitational interactions, this is simply the Einstein–Hilbert action:

$$S_{\mathrm{bulk}} = \frac{M_{\mathrm{Pl}}^{D-2}}{2} \int \mathrm{d}^{D}x \sqrt{-g} R\,, \tag{5.35}$$

where we have defined $M_{\mathrm{Pl}}^{D-2} = 1/(8\pi G)$. To get an intuition of the physics, we will sometimes refer below to a simpler toy model. This consists in a massless scalar field minimally coupled to gravity. Its bulk action is

$$S_{\phi} = \int \mathrm{d}^{D}x \sqrt{-g} \left(-\frac{1}{2} (\partial\phi)^{2} \right)\,. \tag{5.36}$$

ADM mass and metric reconstruction. Recall that, from the point of view of the low-energy theory, the Wilson coefficients in $S_{\rm pp}$ are free parameters. They are either determined experimentally if the UV theory is unknown (as discussed above, the EFT has predictive power, because spacetime and worldline derivatives are small—we can truncate the expansion at the desired order, given a certain level of precision), or they are determined by matching to some microscopic theory. In the following, we will extend the EFT to higher orders in derivatives. However, before getting there, it is instructive to see, as an explicit example, how the matching works for the parameter m in full general relativity. For simplicity, we will focus on the case of non-rotating objects and set the spin to zero.

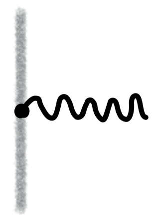

Figure 4: Point-particle gravitational potential at linear order in the number of worldline mass insertions. The vertical line represents the worldline.

The idea is to compute the diagram in figure 4 and compare it with the gravitational potential of a compact source in general relativity. From a path integral perspective, we shall compute the one-point function

$$\langle h_{\mu\nu}(x)\rangle \sim \int \mathcal{D}h\, h_{\mu\nu}(x)\, {\rm e}^{i(S_{\rm bulk}+S_{\rm pp}+S_{\rm GF})}\,, \tag{5.37}$$

where $h_{\mu\nu}$ is defined via

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}, \qquad \kappa \equiv 2M_{\rm Pl}^{1-\frac{D}{2}}\,, \tag{5.38}$$

where and $S_{\rm GF}$ is a gauge-fixing term from the Faddev–Popov procedure. We will work in the background de Donder gauge defined as follows [85]

$$S_{\rm GF} = -\int {\rm d}^4x\sqrt{-\bar{g}}\bar{g}^{\mu\nu}\left[\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha h_{\beta\mu} - \frac{\bar{g}^{\alpha\beta}}{2}\bar{\nabla}_\mu h_{\alpha\beta}\right]\left[\bar{g}^{\rho\sigma}\bar{\nabla}_\rho h_{\sigma\nu} - \frac{\bar{g}^{\rho\sigma}}{2}\bar{\nabla}_\nu h_{\rho\sigma}\right]. \tag{5.39}$$

In this section, we will take $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, although the procedure is general and holds for any background $\bar{g}_{\mu\nu}$. On Minkowski background, the de Donder gauge is defined by the condition

$$\partial^\mu\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = 0, \qquad h \equiv h^\mu_\mu. \tag{5.40}$$

We will compute all connected Feynman diagrams with one external $h_{\mu\nu}(x)$—we shall ignore diagrams containing closed gravitons loop, which do not contribute in the classical limit [81]. The Feynman rules that we need are summarized in figure 5, where

$$G^{\rm dD}_{\mu\nu\rho\sigma}(p^2) = \frac{i}{p^2}P_{\mu\nu\rho\sigma} = -\frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right)\frac{i}{p^2} \tag{5.41}$$

is the graviton propagator.

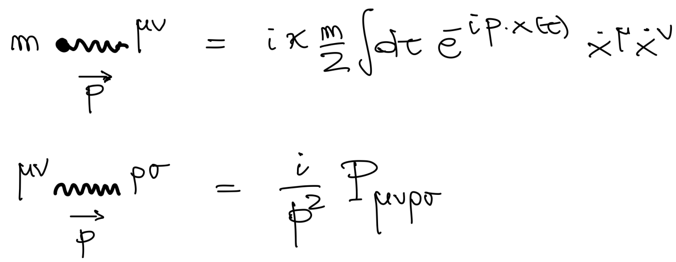

Figure 5: Feynman rules for the calculation of the diagram in figure 4.

At leading order, we have the direct coupling to the point-particle, which sources $h_{\mu\nu}$. The diagram in figure 4 can be calculated as follows:

$$\langle h_{\mu\nu}(x)\rangle = -\frac{\kappa m}{2}\int \mathrm{d}\tau\, \dot{x}^\rho \dot{x}^\sigma \int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\mathrm{e}^{-ik\cdot(x(\tau)-x)}}{k^2} P_{\rho\sigma\mu\nu} \tag{5.42}$$

$$= \frac{\kappa m}{2}\left(\dot{x}_\mu \dot{x}_\nu + \frac{1}{D-2}\eta_{\mu\nu}\right)\int \frac{\mathrm{d}^d \vec{k}}{(2\pi)^d}\frac{\mathrm{e}^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2}\,. \tag{5.43}$$

Using the formula

$$\int \frac{\mathrm{d}^d \vec{k}}{(2\pi)^d}\frac{\mathrm{e}^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2} = \frac{\Gamma(\frac{d}{2}-1)}{(4\pi)^{d/2}}\left(\frac{\vec{x}^2}{4}\right)^{1-\frac{d}{2}}, \tag{5.44}$$

with $r=\sqrt{\vec{x}^2}$, we get

$$\begin{aligned}\langle h_{\mu\nu}(x)\rangle &= \frac{\kappa m}{4(D-2)(D-3)r^{D-3}}\frac{\Gamma(\frac{D-1}{2})}{\pi^{(D-1)/2}}\left(\eta_{\mu\nu}+(D-2)\dot{x}_\mu\dot{x}_\nu\right)\\ &= \frac{\kappa}{2(D-3)r^{D-3}}\frac{r_s^{D-3}}{16\pi G}\left(\eta_{\mu\nu}+(D-2)\dot{x}_\mu\dot{x}_\nu\right),\end{aligned} \tag{5.45}$$

which reproduces the Schwarzschild-Tangherlini metric in isotropic coordinates [103, 104] up to the first non-trivial order in G [105–108] (see [109] for an all-orders reconstruction of the Schwarzschild metric in four dimensions),

$$\mathrm{d}s^2 = \big(\eta_{\mu\nu}+\kappa\langle h_{\mu\nu}(x)\rangle\big)\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1-\left(\frac{r_s}{r}\right)^{D-3}\right)\mathrm{d}t^2 + \left(1+\frac{1}{D-3}\left(\frac{r_s}{r}\right)^{D-3}\right)\mathrm{d}\vec{x}^2 + \dots\,, \tag{5.46}$$

provided one identifies the parameter m with the ADM mass M i.e., in $D=4$,

$$2Gm = 2GM \equiv r_s. \tag{5.47}$$

We stress that, for the identification $m=M$, it is important that the comparison between the IR solution and the full metric linearized in r_s (eq. (5.46)) is done in the same gauge. Alternatively, one can obtain the same result by comparing gauge-invariant quantities in the UV and in the IR.

ADM mass from leading-order worldline equations. Note that the same result at linear order in r_s can be obtained directly from the Einstein equations. The variation of $S_{\rm EFT} = S_{\rm pp} + S_{\rm bulk}$ yields, in $D = 4$, and in the rest frame of the point particle,

$$M_{\rm Pl}^2 G_{\mu\nu}[h] = -m\, \delta^0_\mu \delta^0_\nu\, \delta^{(3)}(\vec{x}), \tag{5.48}$$

where the linearized Einstein tensor is

$$G_{\mu\nu}[h] = -\frac{\kappa}{2}\left[\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \left(\partial^\rho\partial^\sigma h_{\rho\sigma} - \Box h\right)\eta_{\mu\nu}\right]. \tag{5.49}$$

We would like to find the static and spherically-symmetric solutions of (5.48). In the de Donder gauge (5.40), which implies $\partial^\rho\partial^\sigma h_{\rho\sigma} = \frac{1}{2}\Box h$,

$$G_{\mu\nu}[h] = \frac{\kappa}{2}\left(\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h\right). \tag{5.50}$$

Taking the trace of (5.48) with $\eta^{\mu\nu}$, and using (5.50), one finds that $\kappa M_{\rm Pl}^2 \Box h = -2m\, \delta^{(3)}(\vec{x})$. Plugging it back into (5.48), one finds the following equation for $h_{\mu\nu}$:

$$\kappa\, \vec{\nabla}^2 h_{\mu\nu} = -\frac{m}{M_{\rm Pl}^2}\left(2\delta^0_\mu\delta^0_\nu + \eta_{\mu\nu}\right)\delta^{(3)}(\vec{x}), \tag{5.51}$$

where we replaced the d'Alambert operator with the spatial laplacian for time-independent perturbations. Integrating over a sphere of radius r, and using the divergence theorem and the spherical symmetry, one finds the following equation in r: $4\pi\kappa\, r^2 \partial_r h_{\mu\nu} = -\frac{m}{M_{\rm Pl}^2}\left(2\delta^0_\mu\delta^0_\nu + \eta_{\mu\nu}\right)$. Dividing by r and integrating in $\int {\rm d}r$ yields

$$\kappa h_{\mu\nu} = \frac{2Gm}{r}\left(2\delta^0_\mu\delta^0_\nu + \eta_{\mu\nu}\right), \tag{5.52}$$

in agreement with (5.46), provided that the point-particle mass is identified with the ADM mass M of the compact object in isolation.

The matching condition (5.47) can receive corrections on the EFT side in powers of G. One example of correction is, for instance, the self-energy of the static field sourced by the worldline. In the EFT, this type of effects is associated with short-distance singularities, and depends on the regularization scheme. Any discrepancy between the UV behavior of the EFT and the full theory can be compensated by renormalizing the Wilson coefficients by adding local counterterms in $S_{\rm pp}$.

Because the leading-order term in $S_{\rm pp}$ is universal, as argued above, to capture the object's finite size and resolve its internal structure we need to keep the higher-derivative terms in the worldline action. The goal is now to find a set of linearly independent operators that respect the defining symmetries of the problem, and enter at the next order in derivatives. To construct the full set of invariants, as in any EFT, we can take advantage of the worldline equations of motion at previous order to eliminate redundant terms [110]. For example, the leading-order worldline equations of motion (5.24) imply that $a_\mu = 0$ (see eq. (5.25)), so that terms involving the acceleration a_μ can be removed. Similarly, as a consequence of the leading-order

Einstein equations, we can drop operators constructed out of the Ricci curvature tensor [78], i.e.,

$$\int \mathrm{d}\tau\, R(x(\tau)), \qquad\qquad \int \mathrm{d}\tau\, R_{\mu\nu}(x(\tau))u^\mu u^\nu. \tag{5.53}$$

Linear-order redundant operators. To see explicitly how (5.53) can be removed, let's consider a non-rotating body as an example. The point-particle action is (5.20). Using the leading-order Einstein equations, we can express $R_{\mu\nu}$ as a contact term localized on the particle's worldline,

$$R_{\mu\nu} = -\frac{m}{M_{\rm Pl}^{D-2}}\int \mathrm{d}s \frac{\delta^{(D)}(x-x(s))}{\sqrt{-g}}\left(u_\mu u_\nu + \frac{g_{\mu\nu}}{D-2}\right), \tag{5.54}$$

which means that the operators (5.53) can be completely reabsorbed into the definition of the effective parameter m. To be precise, let's perform a field redefinition of the form

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}, \qquad \delta g_{\mu\nu} \equiv \frac{1}{M_{\rm Pl}^{D-2}}\int \mathrm{d}s \frac{\delta^{(D)}(x-x(s))}{\sqrt{-g}}\left(\tilde{c}\, g_{\mu\nu} + \tilde{d}\, u_\mu u_\nu\right), \tag{5.55}$$

with constant $\tilde{c}$ and $\tilde{d}$. The bulk action (5.35) changes as follows:

$$\delta S_{\rm bulk} = \frac{M_{\rm Pl}^{D-2}}{2}\int \mathrm{d}^D x\sqrt{-g}\, G_{\mu\nu}\delta g^{\mu\nu} = \frac{1}{4}\int \mathrm{d}s \left[(\tilde{c}(2-D)+\tilde{d})R + 2\tilde{d}\, R_{\mu\nu}u^\mu u^\nu\right]. \tag{5.56}$$

We can thus choose $\tilde{c}$ and $\tilde{d}$ in such a way to remove the operators (5.53).

Gravitational multipole moments. As a result, the first non-redundant terms that we can add to the worldline at the next order are constructed out of the Riemann tensor, or equivalently the Weyl tensor (see eq. (2.80) for the definition). At linear order in the fields, such operators capture the permanent multipole moments of the object. In full generality, in D spacetime dimension, we can write

$$\begin{aligned} S_{\rm multipoles} = \sum_L \frac{1}{L!}\int \mathrm{d}\tau \Big[&\lambda^{i_1\cdots i_L}_{(L,C_E)}\partial_{(i_1}\cdots\partial_{i_{L-2}}E_{i_{L-1}i_L)_T} + \lambda^{i_1\cdots i_L\, j}_{(L,C_B)}\partial_{(i_1}\cdots\partial_{i_{L-2}}B_{i_{L-1}i_L)_T\, j} \\ &+ \lambda^{i_1\cdots i_L\, j_1 j_2}_{(L,C)}\partial_{(i_1}\cdots\partial_{i_{L-2}}C_{i_{L-1}i_L)_T j_1 j_2}\Big], \end{aligned} \tag{5.57}$$

where we have split the components of the Weyl tensor as

$$E_{ij} \equiv C_{0i0j}\,, \qquad\qquad B_{ijk} \equiv C_{0ijk}\,, \qquad\qquad C_{ijkl}\,. \tag{5.58}$$

Note that the splitting has been done directly into temporal and spatial components in the rest frame of the object. However, one can equivalently define the operators in (5.57) covariantly by means of the projectors $P^\mu_\nu \equiv \delta^\mu_\nu + u^\mu u_\nu$, defined in eq. (5.31). For instance,

$$\lambda^{\mu\nu}_{(2,C_E)}E_{\mu\nu} = \lambda^{\mu\nu}_{(2,C_E)}C_{\mu\rho\nu\sigma}u^\rho u^\sigma. \tag{5.59}$$

Recall that the Weyl tensor has the same symmetries as the Riemann tensor, but in addition it is trace-free, i.e. $C^{\rho}{}_{\mu\rho\nu} = 0$. This implies that the number of independent components of the Weyl tensor in D spacetime dimensions is

$$\frac{D^2(D^2-1)}{12} - \frac{D(D+1)}{2} = \frac{D(D+1)}{2}\left[\frac{D(D-1)}{6} - 1\right]. \tag{5.60}$$

For instance, in $D = 4$ the Riemann tensor has 20 independent components, 10 of which are contained in the Weyl tensor. Moreover, the counting above vanishes for $D = 3$, implying that C_{ikjl} is not independent from C_{0i0j} and C_{0ijk}; in other words, in $D = 4$ the EFT (5.57) contains only $E_{\mu\nu}$ and $B_{\mu\nu\rho}$.

There is a completely analogous story for scalar and vector fields. In those cases, we shall write

$$S^{\text{scalar}}_{\text{multipoles}} = \int \mathrm{d}\tau \left[g\phi + \sum_L \frac{1}{L!}\lambda^{i_1\cdots i_L}_{(L)}\partial_{(i_1}\cdots\partial_{i_L)}\phi\right], \tag{5.61}$$

$$S^{\text{EM}}_{\text{multipoles}} = \int \mathrm{d}\tau \left[eA_\mu v^\mu + \sum_L \frac{1}{L!}\lambda^{i_1\cdots i_L}_{(L,E)}\partial_{(i_1}\cdots\partial_{i_{L-1}}E_{i_L)_T} + \sum_L \frac{1}{L!}\lambda^{i_1\cdots i_L\, j}_{(L,B)}\partial_{(i_1}\cdots\partial_{i_{L-2}}B_{i_L)_T j}\right], \tag{5.62}$$

where ϕ is a scalar field, and E_i and B_{ij} are the electric and magnetic fields defined as

$$E_i \equiv F_{0i}, \qquad B_{ij} \equiv F_{ij}. \tag{5.63}$$

The coupling $g\phi$ represents the object's monopole scalar charge, while $ev^\mu A_\mu$ captures the electric charge of the particle.

In (5.57), the coefficients λ are generic tensors. They can be expressed as functions of the building blocks in the theory (5.26). Naively, there could be an infinite number of combinations of the building blocks, but, at given order in the derivative expansion, only a finite number of them are expected to be linearly independent. For a rotating objects in $D = 4$, one can show that the point-particle action (5.57), at linear order in the metric perturbation $h_{\mu\nu}$, can be rearranged, up to irrelevant boundary terms, as [86, 111]

$$S_{\text{multipoles}} \supset -\frac{\kappa m}{2}\sum_{n=0}^{\infty}\left[\frac{\tilde\lambda_{(n,C_E)}}{(2n)!}\left(-(a^\mu\partial_\mu)^2\right)^n u_\mu u_\nu - \frac{\tilde\lambda_{(n,C_B)}}{(2n+1)!}\left(-(a^\mu\partial_\mu)^2\right)^n u_{(\mu}\epsilon_{\nu)\alpha\beta\gamma}u^\alpha a^\beta\partial^\gamma\right]h^{\mu\nu} + O(h^2) \tag{5.64}$$

where a^μ is related to $S_{\mu\nu}$ via

$$S_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha\beta}u^\alpha a^\beta, \tag{5.65}$$

and where we have suitably redefined the coefficients, $\lambda \mapsto \tilde\lambda$, for practical convenience.

The couplings λ (or, $\tilde\lambda$) capture the gravitational multipole moments of the compact object. Matching (5.64) with the metric of a Kerr black hole in general relativity (in $D = 4$), one finds that $\tilde\lambda_n$ are all unity [111] (see also [112–114]).

Finite-size corrections: conservative effects. We now move on to consider operators in the EFT that are quadratic in the fields and capture effects associated to the finite size of the object. These can be distinguished into two types: conservative and dissipative. Let us start by considering operators that describe conservative effects of the object in the worldline EFT. Later on, we will show how to modify the worldline EFT to include also dissipation in the most general case.

Conservative finite-size effects in the EFT are modeled by local operators on the worldline. Let us start from a simple example in $D = 4$ with zero spin. Since operators involving the Ricci tensor and Ricci scalar can be removed via a field redefinition (see above), the only nontrivial ingredients that we can add are $\{C_{\mu\nu\rho\sigma}, g^{\mu\nu}, \varepsilon^{\mu\nu\rho\sigma}, u^\mu\}$. Using the algebraic properties of the Weyl tensor in $D = 4$ spacetime dimensions, the independent operators that enter at second order in the number of fields and at leading order in derivatives are

$$C_{\mu\alpha\nu\beta}C_{\sigma}{}^{\alpha}{}_{\rho}{}^{\beta}u^\mu u^\nu u^\rho u^\sigma, \qquad C_{\mu\alpha\beta\gamma}C_{\nu}{}^{\alpha\beta\gamma}u^\mu u^\nu, \qquad C_{\lambda}{}^{\mu}{}_{\tau}{}^{\nu}\varepsilon_{\mu\rho\alpha\beta}C^{\alpha\beta}{}_{\nu\sigma}u^\lambda u^\tau u^\rho u^\sigma, \tag{5.66}$$

with two and four u's. The operators (5.66) are the covariant version of

$$E_{ij}E^{ij}, \qquad B_{ijk}B^{ijk}, \qquad E^{ij}\varepsilon_{ikl}B_j{}^{kl}. \tag{5.67}$$

The third operator in eq. (5.66) (or, eq. (5.67)) couples the electric and magnetic components of the Weyl tensor. Such coupling breaks parity/time reversal [78, 115] and is absent in general relativity. Assuming that the underlying full theory respects parity, the first quadratic operators that we can add to the worldline EFT are, to leading order in derivatives,

$$S_{\text{int}} \supset \int \mathrm{d}\tau \left(\lambda_2^{(C_E)}E_{ij}E^{ij} + \lambda_2^{(C_B)}B_{ijk}B^{ijk}\right). \tag{5.68}$$

The couplings $\lambda_2^{(C_E)}$ and $\lambda_2^{(C_B)}$ are commonly called *Love numbers* and capture the leading-order (quadrupolar) deformability of a compact object [116, 117].[18] By dimensional analysis, they are expected to scale, for $L = 2$, as [87]:[19]

$$\lambda_2^{(C_{E,B})} \sim \frac{\mathcal{R}^5}{G}. \tag{5.69}$$

This implies that tidal corrections in a non-relativistic calculation of the orbital dynamics of a compact binary with radius $r \gg r_s$ do not enter until at least the relative order

$$\left(\frac{\mathcal{R}}{r}\right)^5 \sim \left(\frac{\mathcal{R}}{r_s}\right)^5\left(\frac{Gm}{r}\right)^5 \sim \left(\frac{\mathcal{R}}{r_s}\right)^5 v^{10}. \tag{5.70}$$

This means that non-dissipative finite-size effects start affecting the non-relativistic limit of a two-body inspiral from 5PN order.[20] This aspect contributes to the motivation for performing PN calculations to at least 5PN order where such tidal effects start to appear.

Note that tidal deformation effects can however be more pronounced for less compact objects than black holes, due to an enhanced factor $\mathcal{R}/(Gm)$. For instance, for neutron stars $\mathcal{R}/(Gm) \gtrsim O(10)$ [118], and the ratio can be larger for exotic compact objects [119].

[18] The terminology 'Love numbers' refers to the name of the physicist A. E. H. Love, who studied the effects produced by tidal forces on gravitating bodies [116].

[19] Compare (5.68), i.e. schematically $\sim \lambda_2 \int \mathrm{d}\tau(\partial^2 h)^2$, with the point-particle action $m\int \mathrm{d}\tau h$, where h is the gravitational potential $\sim \frac{Gm}{r}$. Thus, $\lambda_2 \sim \mathcal{R}^5/G$, where $\mathcal{R}^{-1}$ represents the cutoff of the EFT. The same comparison yields the $\mathcal{R}^5/r^5$ scaling in (5.70).

[20] Recall that order v^{2n} corresponds to nPN order.

Geodesic deviation. The operators (5.68) are the first ones of an infinite series. One way to see that they correspond to the finite size of the object is to calculate their effect on the motion of a particle moving in a background field. The variation of the point-particle action $S_{\rm pp}$ now includes contributions from the quadratic terms in the curvature:

$$\delta\left[m\int {\rm d}s\right]=\delta\left[\int {\rm d}\tau\left(\lambda_2^{(C_E)}E_{ij}E^{ij}+\lambda_2^{(C_B)}B_{ijk}B^{ijk}\right)\right]\qquad\Rightarrow\qquad a^\mu\neq 0, \tag{5.71}$$

in other words, the particle no longer moves on a geodesic—this should be contrasted with (5.25). This is precisely what happens in gravity when one considers the motion of extended objects in a gravitational field, where geodesic deviation is associated with stretching by tidal forces.

The generalization of (5.68) to arbitrary D-dimensions and higher orders in (spatial) derivatives, for non-rotating objects, is [10, 115]

$$\begin{aligned}S_{\rm int}\supset\sum_{L=1}^{\infty}\frac{1}{2L!}\int{\rm d}\tau\Bigg[&\lambda_L^{(C_E)}\left(\partial_{(a_1}\cdots\partial_{a_{L-2}}E_{a_{L-1}a_L)_T}\right)^2\\&+\frac{\lambda_L^{(C_B)}}{2}\left(\partial_{(a_1}\cdots\partial_{a_{L-2}}B_{a_{L-1}a_L)_Tb}\right)^2+\frac{\lambda_L^{(T)}}{4}\left(\partial_{(a_1}\cdots\partial_{a_{L-2}}C_{a_{L-1}a_L)_Tbc}\right)^2\Bigg].\end{aligned} \tag{5.72}$$

The action (5.72) captures the object's conservative finite-size effects to all orders in the gradients of the gravitational field and zeroth order in the frequency (static limit). Going beyond the static approximation is straightforward: there is an analogous expansion in the number of time derivatives that captures the dynamical response of the object in the adiabatic approximation. For example, at the next-to-leading order in the derivative expansion and quadratic order in the number of fields, we can write

$$S_{\rm int}\supset\lambda_2^{(\dot E)}\int{\rm d}\tau\, u^\rho u^\sigma\nabla_\rho E_{\mu\nu}\nabla_\sigma E^{\mu\nu}=\lambda_2^{(\dot E)}\int{\rm d}\tau\,\dot E_{ij}\dot E^{ij}, \tag{5.73}$$

where in the last equality we went to the rest frame of the point particle. The Wilson coupling $\lambda_2^{(\dot E)}$ is the electric-type dynamical Love number (one can write a similar expression for the magnetic part with coefficient $\lambda_2^{(\dot B)}$), which describes the parity-even leading-order frequency-dependent conservative response. We will come back to the dynamical Love numbers below, when we will write down the most general finite-size effective action including dissipation and spin.

So far, the coefficients $\lambda_2^{(C_E)}$ and $\lambda_2^{(C_B)}$ are generic. Explicit expressions in terms of the microscopic properties of the compact object can be obtained by matching the EFT with a full general-relativity calculation. The most convenient way of proceeding is to match gauge-invariant observables. One example is the quantum mechanical probability amplitude for elastic graviton scattering off the compact object [78, 120, 121]. In this case, one can compute the amplitude in the full theory, by linearizing the Einstein equations around some background field $\bar g_{\mu\nu}$ sourced by the compact object, satisfying the asymptotic boundary conditions

$$h_{\mu\nu}(x) \to \epsilon_{\mu\nu}\,\mathrm{e}^{-ik\cdot x} + \frac{\mathcal{A}_{\mu\nu}}{r}\,\mathrm{e}^{-i\omega(t-r)}, \tag{5.74}$$

where $\epsilon_{\mu\nu}$ is the polarization of the incoming wave and ω is the frequency of the outgoing wave. The scattering amplitude is given by the matrix $\mathcal{A}_{\mu\nu}$ which is gauge invariant, and can be compared with a similar calculation in the EFT. This is of course not the only way of determining the Love number coefficients. Another possibility is to compute the response induced by an external long-wavelength gravitational source. We will come back to this in section 6, where we will perform the calculation of the static response of a compact object in full general relativity and, by performing the matching with the EFT, we will be able to determine the Love number coefficients as functions of the microscopic properties of the object.

Finite-size corrections: including dissipation. In the previous discussion, we have derived an effective theory for the dynamics of an isolated compact object. The EFT is schematically of the form

$$S_{\text{EFT}} = S_{\text{pp}} + S_{\text{bulk}} + S_{\text{multipoles}} + S_{\text{int}}, \tag{5.75}$$

where S_{pp} is the point-particle action in eq. (5.26), S_{bulk} describes the dynamics of fields in the bulk (eq. (5.35)), $S_{\text{multipoles}}$ in (5.57) captures the multipolar structure of the object, while S_{int} contains terms of the form (5.72) that encode information about finite size and tidal response. The ingredients above are all that is required to describe the conservative dynamics, but cannot capture dissipative effects. Such effects are expected to play a relevant role in physical situations: some examples are absorption of gravitational energy by the horizons of black holes, or dissipative tidal deformations of neutron stars due to their internal dynamics.

The way to account for dissipation consists in including a set of internal worldline degrees of freedom. In the following, we will label them as $X(\tau)$. They can effectively absorb energy from external fields and provide a modeling for dissipation in the point particle. The dynamics of the composite object then follows from the action [95, 101]

$$S_{\text{pp}} = \int \mathrm{d}\tau \left(v^\mu p_a(X) e^a_\mu + \frac{1}{2} S^{ab}(X)\Omega_{ab} - \frac{1}{2} e\Big(p_a p^a - L_X(X, e^{-1}\dot{X})\Big) + e\lambda_a S^{ab}(X) p_b(X) \right), \tag{5.76}$$

where we promoted the ingredients in (5.26) to be functions of the additional degrees of freedom X. The internal dynamics is encoded is some Lagrangian L_X whose form will not be needed. The momentum $p^a(X)$ and the spin $S^{ab}(X)$ account for the excitation of the internal degrees of freedom X, and are thus now interpreted as composite operators [101].

To see how the internal degrees of freedom X affect the dynamics of the system, we will integrate them out and obtain an effective action for the kinematic variables $\{x^\mu, e^a_\mu\}$. Since we want to capture dissipative effects, we have to use the in-in formalism. The Schwinger–Keldysh effective action $\Gamma^{\text{in-in}}$ is defined through

$$\mathrm{e}^{i\Gamma^{\text{in-in}}(x_I,(e_I)^a_\mu,e_I)} = \int \mathcal{D}X_1 \mathcal{D}X_2\ \mathrm{e}^{iS[X_1,x_1,(e_1)^a_\mu,e_1] - iS[X_2,x_2,(e_2)^a_\mu,e_2]}\,, \tag{5.77}$$

where $I = 1, 2$. The path integral (5.77) is over two copies of the fields. By construction:

$$\Gamma^{\text{in-in}}(x_I, (e_I)^a_\mu, e_I)\Big|_{x_1=x_2,(e_1)^a_\mu=(e_2)^a_\mu,e_1=e_2} = 0. \tag{5.78}$$

The variation with respect to $\{x^\mu, e^a_\mu\}$ and the einbein e determines the classical equations of motion of the point particle:

$$\frac{\delta}{\delta x^\mu(\tau)}\Gamma^{\text{in-in}}(x_I, (e_I)^a_\mu, e_I)\Big|_{x_1=x_2,(e_1)^a_\mu=(e_2)^a_\mu,e_1=e_2} = 0, \tag{5.79}$$

$$\frac{\delta}{\delta e^a_\mu(\tau)}\Gamma^{\text{in-in}}(x_I, (e_I)^a_\mu, e_I)\Big|_{x_1=x_2,(e_1)^a_\mu=(e_2)^a_\mu,e_1=e_2} = 0. \tag{5.80}$$

Defining the expectation value in the initial state of the internal modes as

$$\langle \mathcal{O}[X_1]\rangle = \int \mathcal{D}X_1\mathcal{D}X_2\, e^{iS[X_1,x_1,(e_1)^a_\mu,e_1]-iS[X_2,x_2,(e_2)^a_\mu,e_2]}\mathcal{O}[X_1], \tag{5.81}$$

for a generic composite operator $\mathcal{O}[X]$, the variation with respect to the einbein e yields

$$\langle p^a p_a - H_X\rangle = 0, \tag{5.82}$$

where the Hamiltonian, in the absence of interactions, is

$$H_X = -\frac{\delta}{\delta e}\Big(eL_X(X, e^{-1}\dot{X})\Big) = \dot{X}\frac{\partial L_X}{\partial \dot{X}} - L_X. \tag{5.83}$$

Without interactions, (5.82) has the interpretation of a mass-shell constraint, which relates the invariant mass $\langle p^a p_a\rangle$ to the initial state of the variable X.

Papapetrou–Mathison–Dixon equations. Let us focus on the free point-particle action (5.76), which we shall rewrite as

$$S_{\text{pp}} = \int \mathrm{d}\tau \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}p_a e^a_\mu + \frac{1}{2}S_{ab}g^{\mu\nu}e^a_\mu\frac{D}{D\tau}e^b_\nu - \frac{1}{2}e\Big(p_a p^a - L_X(X, e^{-1}\dot{X})\Big) + e\lambda_a S^{ab}(X)p_b(X)\right). \tag{5.84}$$

The variation with respect to the kinematic variables $(x^\mu, p_a, e^a_\mu, S_{ab}, \lambda_a, e)$ yields the Papapetrou–Mathison–Dixon equations of motion [122–124]. For instance, taking the variation with respect to e^a_μ (and holding the metric fixed):

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}p_a + \frac{1}{2}S_{ab}g^{\mu\alpha}\frac{D}{D\tau}e^b_\alpha + \frac{1}{2}\frac{D}{D\tau}(S_{ab}e^b_\alpha g^{\alpha\mu}) = 0. \tag{5.85}$$

Contracting with $e^a_\beta g^{\beta\nu}$ and rewriting the last term as $\frac{1}{2}e^a_\beta g^{\beta\nu}\frac{D}{D\tau}(S_{ab}e^b_\alpha g^{\alpha\mu}) = \frac{1}{2}\frac{D}{D\tau}S^{\nu\mu} + \frac{1}{2}S_{ab}e^a_\alpha g^{\alpha\mu}\frac{D}{D\tau}(e^b_\beta g^{\beta\nu})$, we get, after taking the antisymmetric combination:

$$\frac{D}{D\tau}S^{\mu\nu} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}p^\nu - \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}p^\mu. \tag{5.86}$$

Analogously, the variation with respect to $x^\mu(\tau)$ (and holding the tetrad fixed) yields [79]

$$\frac{D}{D\tau}p^\mu = -\frac{1}{2}R^\mu{}_{\nu\rho\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}S^{\rho\sigma}, \tag{5.87}$$

which generalizes (5.25) for nonzero spin. The right-hand side of (5.87) corresponds to the usual Papapetrou–Mathison–Dixon force of a spinning point particle, in the absence of other interactions.

In the absence of dissipation, the variation with respect to p^μ and $S^{\mu\nu}$ gives a relation between $(p^\mu, S^{\mu\nu})$ and $(\dot{x}^\mu, \Omega^{\mu\nu})$. However, in the presence of dissipation, the momentum variables depend on the internal degrees of freedom X and cannot be obtained in a model-independent. In other words, one needs in principle to know the dynamics of X, i.e. $L_X(X, e^{-1}\dot{X})$. In some cases, this can be sidestepped if one knows the microscopic theory, and one can obtain the relation between $S^{\mu\nu}$ and $\Omega^{\mu\nu}$ by performing an explicit matching [101]. On the other hand, the relation between $\dot{x}^\mu$ and p^μ follows in general from (5.86), (5.87) and the constraint $S^{ab}p_b = 0$. In fact, from $\frac{D}{D\tau}(S^{\mu\nu}p_\nu) = p_\nu \frac{D}{D\tau}S^{\mu\nu} + S^\mu{}_\nu \frac{D}{D\tau}p^\nu$, and from eqs. (5.86) and (5.87), it follows that [101]

$$p^2 \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} - p^\mu p_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} - \frac{1}{2} R_{\nu\lambda\rho\sigma} \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} S^{\rho\sigma} S^{\mu\nu} = 0. \tag{5.88}$$

Note that, in the absence of interactions, for an object at rest one recovers that p^μ is proportional to $\mathrm{d}x^\mu/\mathrm{d}\tau$. Then, the equations of motion imply that

$$p^2 = M^2, \qquad S^2 \equiv \frac{1}{2} S^{\mu\nu} S_{\mu\nu} \tag{5.89}$$

are conserved along the worldline, i.e.,

$$\frac{D}{D\tau} p^2 = 0, \qquad \frac{D}{D\tau} S^2 = 0. \tag{5.90}$$

The free Papapetrou–Mathison–Dixon equations will be modified by the presence of interactions and finite-size operators in the EFT. Similarly to (5.71), tidal couplings will modify the equations of motion for the kinetic variables. In particular, including dissipation, we have in general

$$\frac{D}{D\tau} p^2 \neq 0, \qquad \frac{D}{D\tau} S^2 \neq 0, \tag{5.91}$$

where the right-hand sides are expected to involve the in-in expectation value of the tidal operators in the state of the internal modes.

To be concrete, on top of the point-particle action (5.76), let's focus on the leading-order tidal coupling operators. For illustrative purposes, let's consider the electric sector only. We shall thus write

$$S_{\text{int}} \supset \int \mathrm{d}\tau \, e \, Q_{ij}^{(E)}(X, \tau) E^{ij}(x(\tau)), \tag{5.92}$$

where $Q_{ij}^{(E)}(X, \tau)$ is a composite operator that depends on the internal degrees of freedom X. Diagrammatically, the interaction (5.92) can be expressed as in figure 6.

From a bottom-up perspective, we do not have in general access to the explicit form of $Q_{ij}^{(E)}(X, \tau)$. Nevertheless, we can still analyze the structure of its correlation functions. In practice, what we will do is to parametrize them with some unknown coefficients, which carry all the information about the unknown

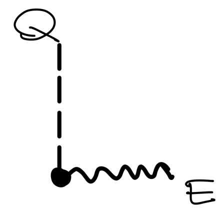

Figure 6: Interaction vertex between the worldline and the Q and E fields.

UV physics, and fix them by performing a matching with some explicit models. For example, imagine that we are interested in computing the one-point function $\langle Q^{(E)}_{ij}(\tau)\rangle_{\text{in-in}}$. For weak external fields, we expect $\langle Q^{(E)}_{ij}(\tau)\rangle_{\text{in-in}}$ from linear response theory to be of the form

$$\langle Q^{(E)}_{ij}(\tau)\rangle_{\text{in-in}} = \int \mathrm{d}\tau' G_R^{ij|kl}(\tau-\tau')E_{kl}(x(\tau')), \tag{5.93}$$

where $E_{kl}(x(\tau'))$ plays the role of the external source, while $G_R^{ij|kl}(\tau-\tau')$ is the retarded Green's function, defined as

$$G_R^{ij|kl}(\tau-\tau') = -i\langle[Q^{(E)ij}(\tau), Q^{(E)kl}(\tau')]\rangle\Theta(\tau-\tau'), \tag{5.94}$$

where $\Theta(\tau)$ is the Heaviside step function. Since $G_R^{ij|kl}(\tau-\tau')$ is a real quantity, it satisfies

$$\left[G_R^{ij|kl}(-\omega)\right]^* = G_R^{ij|kl}(\omega) \tag{5.95}$$

for real frequencies. Therefore, it must be that $\mathrm{Re}[G_R^{ij|kl}(\omega)]$ is an even function of ω on the real ω-axis, while $\mathrm{Im}[G_R^{ij|kl}(\omega)]$ is odd. In general we do not know how G_R depends on the microscopic physics of the system, but we know that, whatever the UV theory is, it must be analytic for $\mathrm{Im}\,\omega \geq 0$ [101]. If the typical time scale of the external perturbing field is long compared with the time scale of the internal dynamics, we can work in an adiabatic approximation. From a low-energy perspective, we shall thus expand the retarded Green's function in powers of the frequency. The most general parametrization that is compatible with these requirements is, in frequency space,

$$G_R^{ij|kl}(\omega) = \lambda_0^{ij|kl} + i(r_s\omega)\lambda_1^{ij|kl} + (r_s\omega)^2\lambda_2^{ij|kl} + \cdots\,. \tag{5.96}$$

Let us comment on the following aspects:

- The tensorial structure in the expressions above, e.g. (5.92), stems from the fact that, for rotating objects, one can construct different combinations involving the spin [99, 101, 121]. In other words, the idea is to find a basis of linearly independent tensors, which we can use to write the most general parametrization of $\lambda_n^{ij|kl}$.[21] For instance, the leading order coefficient can be written in full generality, in $D=4$, as [121]

$$\lambda_0^{ij|kl} = \Lambda_{\omega^0,S^0}\delta^{(i}_{(k}\delta^{j)_T}_{l)_T} + H_{\omega^0,S^1}S^{(i}{}_{(k}\delta^{j)_T}_{l)_T} + \Lambda_{\omega^0,S^2}s^{(i}s_{(k}\delta^{j)_T}_{l)_T} + H_{\omega^0,S^3}s^{(i}s_{(k}S^{j)_T}{}_{l)_T} + \Lambda_{\omega^0,S^4}s^{(i}s_{(k}s^{j)_T}s_{l)_T}, \tag{5.97}$$

[21] We are assuming to be in the frame where the point particle has zero velocity, so that indices here and below are purely spatial.

where

$$s^i = \frac{1}{2}\varepsilon^{ijk} S_{jk} \tag{5.98}$$

is the Pauli–Lubanski spin vector, and where Λ and H are some coefficients. Similar expansions can be written down for the other tensors in (5.96), e.g.,

$$\lambda_1^{ij|kl} = H_{\omega^1,S^0}\delta^{(i}_{(k}\delta^{j)_T}_{l)_T} + \Lambda_{\omega^1,S^1} S^{(i}{}_{(k}\delta^{j)_T}_{l)_T} + H_{\omega^1,S^2} s^{(i} s_{(k}\delta^{j)_T}_{l)_T} + \Lambda_{\omega^1,S^3} s^{(i} s_{(k} S^{j)_T}{}_{l)_T} + H_{\omega^1,S^4} s^{(i} s_{(k} s^{j)_T} s_{l)_T}, \tag{5.99}$$

$$\lambda_2^{ij|kl} = \Lambda_{\omega^2,S^0}\delta^{(i}_{(k}\delta^{j)_T}_{l)_T} + H_{\omega^2,S^1} S^{(i}{}_{(k}\delta^{j)_T}_{l)_T} + \Lambda_{\omega^2,S^2} s^{(i} s_{(k}\delta^{j)_T}_{l)_T} + H_{\omega^2,S^3} s^{(i} s_{(k} S^{j)_T}{}_{l)_T} + \Lambda_{\omega^2,S^4} s^{(i} s_{(k} s^{j)_T} s_{l)_T}, \tag{5.100}$$

$$\vdots$$

an so on for higher powers of ω [121, 125]. Analogous expressions hold in higher spacetime dimensions. Note that, in the case of non-rotating objects, everything boils down to the symmetric-trace-free product of Kronecker symbols.

In the expressions above, we used two different symbols for the tidal coefficients, depending on the numbers of ω and spin factors at each order: Λ denotes terms associated with conservative effects, while H refers to dissipative terms ('H' stands for tidal 'heating'). In fact:

- In the previous parametrization of the response, the operators $D/D\tau$, S_{ij} and s_i are odd under time reversal. All terms in (5.96) that respect time-reversal invariance can be reabsorbed into local contact terms on the worldline and contribute to the conservative tidal deformation. For instance, for non-rotating objects, such terms are all those that are even in ω (to leading order in derivatives and in $D = 4$, we already wrote them in eq. (5.68)): in particular, Λ_{ω^0,S^0} corresponds to the static Love numbers, while $\Lambda_{\omega^{2n},S^0}$ for $n \geq 1$ represent the dynamical Love numbers.

- Instead, all terms that contain a total odd number of ω, S_{ij} and s_i break time-reversal invariance and correspond to tidal dissipation (or, equivalently, tidal heating). As such, they cannot be written in the form of local operators on the worldline. For non-rotating objects, such operators are defined by an odd number of ω in the adiabatic expansion (5.96).

- Similar considerations hold in the presence of the magnetic tidal field B_{ijk}. The only difference that one should keep in mind is that under a time-reversal transformation B is odd, as opposed to E which is even [121, 125].

A summary of the main conservative and dissipative coefficients in the worldline EFT is given in table 1 (see [121, 125] for further details). We also report the PN order at which each coefficient (in the electric sector) starts contributing [79]. The first two rows correspond to the conservative sector, while the last three contain the leading dissipation coefficients. Note that, within each sector, increasing the power of ω by one is tantamount to adding a +1.5PN: this stems from the fact that (see eq. (5.1)):

$$\omega \sim \frac{v}{r} \sim v^3. \tag{5.101}$$

Response coefficients	Notation	Start contributing at PN order
Tidal (static) Love numbers	$\Lambda^E_{\omega^0,S^{2n}}$	5
Dynamical tidal Love numbers	$\Lambda^E_{\omega^1,S^{2n+1}}$ $\Lambda^E_{\omega^2,S^{2n}}$	6.5 8
LO (static) tidal dissipation numbers	$H^E_{\omega^0,S^{2n+1}}$	2.5
NLO tidal dissipation numbers	$H^E_{\omega^1,S^{2n}}$	4
NNLO tidal dissipation numbers	$H^E_{\omega^2,S^{2n+1}}$	5.5

Table 1: Leading-order (LO) and next-to-leading-order (NLO) conservative and dissipative tidal response coefficients in the worldline EFT, including the PN order at which they start contributing. Note that we introduced an extra superscript E on Λ and H to emphasize that the PN order in the rightmost column refers to the electric sector E: magnetic tidal dissipation effects start appearing at 1PN order higher than their electric counterparts (this essentially follows from the relative velocity suppression, $B \sim vE$) [125].

This explains the increasing PN counting from top to bottom in the rightmost column in table 1. We stress that, for rotating bodies, the PN counting is implicitly done assuming large spin [79].

The generalization to higher orders in spatial gradients is straightforward; it is enough to add more indices. Note that adding spatial derivatives is equivalent to adding extra powers of $\frac{1}{r}$, i.e. increasing the PN order at which the effect starts contributing.[22] The interaction terms, capturing the linear (conservative and dissipative) response of the system, take now the form

$$S_{\rm int} \supset \int \mathrm{d}\tau \, e \, Q^{i_1\cdots i_L}(X)\, C^{(s)}_{i_1\cdots i_L} \,, \tag{5.102}$$

where $C^{(s)}_{i_1\cdots i_L}$ stands schematically for external probe fields: they can belong to the scalar, vector and tensor gravitational sectors, but can capture also scalar and spin-1 fields. For example, for a scalar field ϕ, one has

$$C^{(0)}_{i_1\cdots i_L} = \partial_{(i_1} \cdots \partial_{i_L)_T} \phi \,, \tag{5.103}$$

while for gravity the possible combinations, built out of the Weyl tensor, are the ones written e.g. in eq. (5.57):

$$\partial_{(i_1} \cdots \partial_{i_{L-2}} E_{i_{L-1} i_L)_T} , \qquad \partial_{(i_1} \cdots \partial_{i_{L-2}} B_{i_{L-1} i_L)_T\, j} , \qquad \partial_{(i_1} \cdots \partial_{i_{L-2}} C_{i_{L-1} i_L)_T j_1 j_2} . \tag{5.104}$$

Integrating out the X fields, we will find an effective action (see section 6 for further details) where the two-point function of Q appears. Since we are interested in the classical limit, the relevant two-point function is the causal retarded Green's function [101], which at generic order in the multipolar expansion takes the form

$$G_R^{i_1\cdots i_L | j_1 \cdots j_{L'}}(\tau - \tau') = -i\Theta(\tau-\tau')\langle [Q^{i_1\cdots i_L}(\tau), Q^{j_1\cdots j_{L'}}(\tau')]\rangle \,, \tag{5.105}$$

[22] Schematically, one more spatial derivative on an EFT operator brings a +1PN with respect to the previous order operator in the same conservative/dissipative sector.

which we will expand as

$$G_R^{i_1\cdots i_L|j_1\cdots j_{L'}}(\omega) = \lambda_0^{i_1\cdots i_L|j_1\cdots j_{L'}} + i(r_s\omega)\lambda_1^{i_1\cdots i_L|j_1\cdots j_{L'}} + (r_s\omega)^2\lambda_2^{i_1\cdots i_L|j_1\cdots j_{L'}} + \cdots \tag{5.106}$$

where the coefficients admit a decomposition in terms of conservative and dissipative parameters as the one in table 1.

Note that, at quadratic order in the EFT action, the breaking of spherical symmetry in the presence of nonzero spin implies possible mixing effects between terms with a different number of spatial gradients. One example is the quadrupole-octupole mixing, which is schematically of the form $E \cdot S \cdot \partial B$ [121].

Beyond linear-response approximation and nonlinear Love numbers. All the previous discussion admits a natural extension to higher orders in perturbation theory: the EFT provides a systematic framework to describe nonlinear effects and nonlinear deformation of compact objects. In particular, we can describe the response beyond the linear approximation (see, e.g., [81, 126–130]). For example, at leading order in derivatives, in the purely conservative electric (even) sector, one can write the following nonlinear interaction terms [128, 129]

$$S_{\rm int} \supset \int \mathrm{d}\tau\, e \sum_{n=1}^{\infty} \left[\lambda_n^{(E)} E_{\mu_1}{}^{\mu_2} \cdots E_{\mu_{n+1}}{}^{\mu_1} + \ldots\right]. \tag{5.107}$$

In a similar fashion, one can describe dissipative effects. Terms on the worldline like these ones reflect the nonlinear couplings between gravity and the UV modes that have been integrated out. In eq. (5.107), the coefficients $\lambda_n^{(E)}$ correspond to the nonlinear Love numbers.

Summary. In the worldline EFT, finite-size effects—like the tidal deformability of the object—are captured by coefficients of higher-dimensional operators. The operators are in general composite objects that depend on the internal degrees freedom and can account for dissipation in the system. As opposed to the dissipative response, conservative response is encoded in local operators on the worldline, whose coefficients are the Love numbers. One main advantage in using the EFT to define the linear response of the system is that the Love numbers are coupling constants of gauge invariant operators. As such, they are independent of the choice of coordinates.

6 Tidal deformability and induced linear response of black holes

In the previous section, we introduced the worldline EFT and defined the conservative and dissipative response coefficients in the EFT. In this section, we want to determine those coefficients in terms of the microscopic properties of the compact object by performing an explicit matching. We will first solve the full Einstein equations in general relativity in the small-frequency limit by requiring that the solution approaches a tidal field profile at infinity. Using the ingredients of the previous section, we will next perform an analogous calculation in the worldline EFT and compute the response field. Finally, we will compare the response field solutions and match the effective couplings. Before doing so, it is instructive to consider an analogy with a more familiar case in electromagnetism.

Warm-up example. Let's consider a dielectric sphere of radius $\mathcal{R}$ in electromagnetism and recall the standard calculation of the electric polarization in the presence of an external electric field [131]. In the limit of static fields, the Laplace equation for the potential is $\vec{\nabla}^2\Phi = 0$. After imposing regularity at the center of the sphere, the solution inside the dielectric is

$$\Phi_{\rm int} = \sum_L A_L r^L P_L(\cos\theta), \tag{6.1}$$

in terms of the Legendre polynomials P_L. Outside the dielectric, the most general solution is of the form

$$\Phi_{\rm ext} = \sum_L \left[B_L r^L + C_L r^{-L-1}\right] P_L(\cos\theta). \tag{6.2}$$

The boundary condition at $r=\infty$ fixes B_L, which will match the amplitude of the external field, while A_L and C_L are determined by matching conditions across the surface of the dielectric:

$$\frac{\partial}{\partial\vec{\theta}}\Phi_{\rm int} = \frac{\partial}{\partial\vec{\theta}}\Phi_{\rm ext}, \qquad \epsilon\frac{\partial}{\partial r}\Phi_{\rm int} = \epsilon_0\frac{\partial}{\partial r}\Phi_{\rm ext}, \tag{6.3}$$

in terms of the vacuum and dielectric permittivities ϵ_0 and ϵ. The solutions are

$$A_L = B_L\frac{\epsilon_0(2L+1)}{(L+1)\epsilon_0 + L\epsilon}\,, \qquad C_L = B_L\frac{L(\epsilon_0-\epsilon)\mathcal{R}^{2L+1}}{(L+1)\epsilon_0 + L\epsilon}\,. \tag{6.4}$$

Outside the sphere, the potential is equivalent to the applied field plus the potential of an electric multipole moment given by the coefficient of r^{-L-1}. In generic D spacetime dimensions, the two falloffs correspond to r^L and $r^{-(L+D-3)}$.

A similar calculation can in principle be done in gravity. Instead of Maxwell equations, one would solve the Einstein equations, impose suitable matching conditions at the surface of the object and extract the coefficients of the induced $r^{-(L+D-3)}$ tail at large distances. However, for gravity the situation is slightly more delicate. In fact, as opposed to electromagnetism, freedom associated with the choice of coordinates and nonlinearities in general relativity may introduce ambiguities in the definition of the response coefficients. We will give some explicit examples below. One way to get around these issues is to define the tidal response of the object in the worldline EFT, as we have seen in the previous section.

In the following, let's first see how the response field is computed in full general relativity. We will later match the result with the worldline EFT and extract the response coefficients.

6.1 Linear response fields: full theory calculation

Conservative response. Let us start by focusing on the conservative response at zero ω for a non-rotating black hole. For simplicity, we will consider a scalar perturbation [10, 97]. The generalization to gravitational perturbations is conceptually straightforward. For Schwarzschild black holes, the calculation of the static Love numbers was originally obtained in [132–134] (see also [10, 97, 120, 135]). It was later generalized to the case of Kerr black holes in [99, 136–138]. Instead, the Love numbers of charged, non-rotating black holes in Einstein–Maxwell theory in four spacetime dimensions have been studied in [139–141].

Let us take the Klein–Gordon equation $\Box\phi = 0$ for a massless scalar field ϕ. Decomposing in scalar spherical harmonics as

$$\phi(x) = \sum_{L,M} \Psi(t,r) r^{\frac{2-D}{2}} Y_L^M(\theta), \tag{6.5}$$

the equation for Ψ takes the Schrödinger-like form (2.67). In the zero-frequency limit, it reduces to

$$f\Psi'' + f'\Psi' - \left(\frac{L(L+D-3)}{r^2} + f'\frac{D-2}{2r} + f\frac{(D-4)(D-2)}{4r^2} \right) \Psi = 0\,, \tag{6.6}$$

where we have reverted from the tortoise coordinate back to the original radial coordinate r. It is convenient to introduce the dimensionless radial variable

$$x \equiv \left(\frac{r_s}{r}\right)^{D-3}, \tag{6.7}$$

such that the black hole horizon is now located at $x = 1$, while spatial infinity corresponds to $x = 0$. In addition to this coordinate change, we also perform the field redefinition

$$u(x) \equiv x^{-\frac{D+2L-4}{2(D-3)}} \Psi(r(x))\,, \tag{6.8}$$

which recasts eq. (6.6) as a hypergeometric equation in the standard form:

$$x(1-x)u''(x) + \big[c - (a+b+1)x\big]u'(x) - a\,b\,u(x) = 0\,, \tag{6.9}$$

where the parameter values are given by

$$a = \hat{L} + 1\,, \qquad b = \hat{L} + 1\,, \qquad c = 2\hat{L} + 2\,, \qquad \text{with} \qquad \hat{L} \equiv \frac{L}{D-3}. \tag{6.10}$$

Notice that the parameters a, b and c satisfy the condition $a+b-c = 0$. The benefit of these transformations is that the hypergeometric equation is extremely well-studied, and therefore the solutions of interest are readily available in the literature.

The differential equation (6.9) is a second-order equation, so we require two boundary conditions to specify completely a solution. On physical grounds, the first requirement we will impose is that our solution be regular at the black hole horizon, i.e., at $x = 1$. The second boundary condition fixes instead the normalization of the growing mode solution at radial infinity, i.e., at $x = 0$. We can then read off the induced sub-leading fall-off at radial infinity, which captures the linear response to the externally applied field. Note that the overall normalization of the solution at infinity is formally a boundary condition, but it does not affect the ratio of the growing and decaying modes at infinity, which is ultimately what we are interested in.

We will derive the solution for generic values of the parameter $\hat{L}$. In principle, one has to be careful, because for certain values of the parameters (6.10) the hypergeometric equation becomes degenerate, i.e. the two standard solutions are no longer linearly independent. This happens in two cases: when $\hat{L}$ is either integer or semi-integer. In the end, this will not be a problem because the limit of $\hat{L}$ being integer or semi-integer is smooth [10, 97]. In other words, we can safely compute the Love numbers for generic $\hat{L}$ and take the integer or semi-integer limit at the end. See Refs. [10, 97] for details. Note that in $D = 4$ $\hat{L}$ is always an integer number: this procedure will allow us also to stress an important difference between Love numbers in $D = 4$ versus $D > 4$.

When $\hat{L}$ is neither integer nor half-integer (such that all the parameters a, b, $c - a$, $c - b$, and c are non-integer), the two linearly independent solutions to (6.9) are [142–144]

$$u_1(x) = {}_2\mathsf{F}_1\left[\begin{array}{c}\hat{L}+1, \quad \hat{L}+1\\ 2\hat{L}+2\end{array}\middle| x\right] \quad \text{and} \quad u_5(x) = x^{-2\hat{L}-1}\, {}_2\mathsf{F}_1\left[\begin{array}{c}-\hat{L}, \quad -\hat{L}\\ -2\hat{L}\end{array}\middle| x\right]. \tag{6.11}$$

This basis of solution is particularly natural because it corresponds to the two linearly independent falloffs near $x = 0$. Using standard hypergeometric identities, one finds that the particular linear combination that remains finite at the horizon ($x = 1$) is given by [10]

$$u(x) = A\left(\frac{\Gamma(-2\hat{L}-1)}{\Gamma(-\hat{L})^2}\, {}_2\mathsf{F}_1\left[\begin{array}{c}\hat{L}+1, \quad \hat{L}+1\\ 2\hat{L}+2\end{array}\middle| x\right] + \frac{\Gamma(2\hat{L}+1)}{\Gamma(\hat{L}+1)^2} x^{-2\hat{L}-1}\, {}_2\mathsf{F}_1\left[\begin{array}{c}-\hat{L}, \quad -\hat{L}\\ -2\hat{L}\end{array}\middle| x\right]\right), \tag{6.12}$$

with A an overall normalization constant. Note that we can use the connection formula

$$\begin{aligned}{}_2\mathsf{F}_1\left[\begin{array}{c}a, \quad b\\ a+b-c+1\end{array}\middle| 1-x\right] = {}& \frac{\Gamma(1-c)\Gamma(a+b-c+1)}{\Gamma(a-c+1)\Gamma(b-c+1)}\, {}_2\mathsf{F}_1\left[\begin{array}{c}a, \quad b\\ c\end{array}\middle| x\right]\\ &+ \frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)} x^{1-c}\, {}_2\mathsf{F}_1\left[\begin{array}{c}a-c+1, \quad b-c+1\\ 2-c\end{array}\middle| x\right],\end{aligned} \tag{6.13}$$

to rewrite eq. (6.12) in a more compact form as [97]

$$u(x) = A\, {}_2\mathsf{F}_1\left[\begin{array}{c}\hat{L}+1, \quad \hat{L}+1\\ 1\end{array}\middle| 1-x\right], \tag{6.14}$$

which is manifestly regular in the limit $x \to 1$, since hypergeometric functions are normalized in such a way that ${}_2\mathsf{F}_1(a, b; c; 0) = 1$.

In order to extract the Love numbers, we expand the solution (6.14) around $x = 0$ to find

$$u(x \to 0) \simeq A\left(\frac{\Gamma(-2\hat{L}-1)}{\Gamma(-\hat{L})^2} + \cdots + \frac{\Gamma(2\hat{L}+1)}{\Gamma(\hat{L}+1)^2} x^{-2\hat{L}-1} + \ldots\right), \tag{6.15}$$

where we are keeping only the contributions corresponding to the two linearly independent falloffs at infinity, which are the ones relevant for the calculation of the response.

In terms of the radial coordinate, eq. (6.15) takes the form

$$u(r\to\infty)\simeq A\left(\frac{r}{r_s}\right)^{L+D-3}\left(\frac{\Gamma(2\hat{L}+1)}{\Gamma(\hat{L}+1)^2}\left(\frac{r}{r_s}\right)^{L}+\cdots+\frac{\Gamma(-2\hat{L}-1)}{\Gamma(-\hat{L})^2}\left(\frac{r_s}{r}\right)^{L+D-3}+\ldots\right), \tag{6.16}$$

with the first term to be interpreted as an external tidal field with overall amplitude A, while the second term encodes the response of the system. As expected for weak perturbations, the response is linear in the magnitude of the external field. Note that this is the analog of eq. (6.2) in the dielectric example above.

We are interested in the ratio between the coefficient of the induced $r^{-(L+D-3)}$ tail of the solution and the r^L tidal component, measured in units of r_s^{2L+D-3} [10, 97]:

$$k_{\text{scalar}}=\frac{\Gamma(-2\hat{L}-1)}{\Gamma(-\hat{L})^2}\frac{\Gamma(\hat{L}+1)^2}{\Gamma(2\hat{L}+1)}=\frac{2\hat{L}+1}{2\pi}\frac{\Gamma(\hat{L}+1)^4}{\Gamma(2\hat{L}+2)^2}\tan(\pi\hat{L})\,. \tag{6.17}$$

Let us make some observations:

- The coefficients (6.17) vanish whenever $\hat{L}$ is integer. In particular, they vanish in $D=4$ for all multipoles. Similar expressions can be found for higher spins and, in particular, gravitational perturbations (e.g., [10, 97, 115]). This is associated to the well-known property that the static tidal Love numbers of asympotically flat Schwarzschild black holes vanishes in $D=4$ general relativity [132–134], as we shall explicitly see by performing the matching below in (6.56).

- The expression (6.17) is formally divergent when $\hat{L}$ is semi-integer. One can use

$$\Gamma(-n+\epsilon)=\frac{(-1)^n}{n!\,\epsilon}+O(\epsilon^0), \tag{6.18}$$

  with $n=2\hat{L}\in\mathbb{N}$ and introduce a regularization prescription to subtract the $1/\epsilon$ term [97], or rederive more carefully the hypergeometric solution for semi-integer $\hat{L}$ paying attention to the degeneracy [10]. In fact, when $\hat{L}$ is half-integer the two solutions (6.11) cease to be linearly independent. Translating $\hat{L}$ back into the parameters a,b,c using (6.10), $\hat{L}$ being half-integral implies that a, b, $c-a$ and $c-b$ are non-integer, while c takes positive integer values. Using this information we can use standard results to find a new basis of solutions [143]

$$u_1(x)={}_2\mathsf{F}_1\left[\begin{matrix}\hat{L}+1,\quad \hat{L}+1\\ 2\hat{L}+2\end{matrix}\,\middle|\,x\right]\quad\text{and}\quad u_2(x)={}_2\mathsf{F}_1\left[\begin{matrix}\hat{L}+1,\quad \hat{L}+1\\ 1\end{matrix}\,\middle|\,1-x\right]. \tag{6.19}$$

  Note that the first solution $u_1(x)$ contains a logarithmic divergence of the form $\log(1-x)$ around $x=1$. On the other hand $u_2(x)$ is finite as $x\to1$ and therefore it is the solution describing the physical scalar perturbations around a Schwarzschild black hole. In particular, since $c=1,2,3,\cdots$ and $a,b\neq c-1,c-2,\cdots,0,-1,-2,\cdots$, one can infer its asymptotic expansion in the neighborhood of $x=0$ via the formula

$$
{}_2\mathsf{F}_1\left[\begin{array}{c} a,\quad b \\ 1+a+b-c \end{array}\Big| 1-x\right] = {}_2\mathsf{F}_1\left[\begin{array}{c} a,\quad b \\ c \end{array}\Big| x\right]\log x - \sum_{n=1}^{c-1}\frac{(c-1)!(n-1)!}{(c-n-1)!(1-a)_n(1-b)_n}(-x)^{-n}
$$
$$
+\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\big[\psi(a+n)+\psi(b+n)-\psi(1+n)-\psi(c+n)\big]x^n\,, \tag{6.20}
$$

where $\psi(x)\equiv\Gamma'(x)/\Gamma(x)$ is the digamma function. This asymptotic expansion is of a drastically different form than eq. (6.15). In fact, keeping in (6.20) only the leading term and the one that scales like $x^{-2\hat{L}-1}$, as we did in eq. (6.15), and substituting in (6.10) for a,b,c one finds:

$$
u_2(x)\simeq\log x+\cdots+(-1)^{2\hat{L}}(2\hat{L})!(2\hat{L}+1)!\frac{\Gamma(-\hat{L})^2}{\Gamma(\hat{L}+1)^2}x^{-2\hat{L}-1}+\cdots\,. \tag{6.21}
$$

An important difference compared to the case studied above is that (6.21) does not consist only of powers of x, but contains also a logarithmic divergence as $x\to 0$. This logarithm can be understood as a classical running of the value of the induced response [97]. In more detail, we can take the ratio of the two fall-offs in (6.21) to define the dimensionless response (in units of r_s^{2L+D-3}):

$$
k_{\text{scalar}}=\frac{(-1)^{2\hat{L}}(D-3)\Gamma(\hat{L}+1)^2}{(2\hat{L})!(2\hat{L}+1)!\Gamma(-\hat{L})^2}\log\left(\frac{r_0}{r}\right)\qquad\qquad(\text{half-integer }\hat{L}). \tag{6.22}
$$

Here we have only recorded the coefficient of the logarithmic term in the ratio of fall-offs in (6.21). This is because only these terms are unambiguous. The dependence on r—which we can think of as the distance at which we measure the response of the system—is an example of classical renormalization group (RG) running. The length scale r_0 is a renormalization scale to be fixed by experiments, but that on physical grounds we expect to be of $\mathcal{O}(r_s)$. In [97], the Love numbers in this degenerate case are obtained from the general expression (6.17) by taking the limit of half-integer values for $\hat{L}$. This limit is singular, but it is possible to isolate a finite contribution by a suitable (classical) renormalization procedure that removes the divergent piece. As expected, the value (6.22) has precisely the same logarithmic term as in [97], but differs in the finite terms.

Dissipative response. We have seen above that the induced field of a non-rotating black hole in generic spacetime dimension has no imaginary part. As a result, this implies that there is no dissipation, as expected. Let us now include frequency corrections. For simplicity, we will compute the solution at linear order in ω and set the spin of the black hole to zero [145–149]. We will comment later on about Kerr black holes.

After performing the usual decomposition

$$
\phi(x)=\sum_{L,M}\Psi(t,r)r^{\frac{2-D}{2}}Y_L^M(\theta), \tag{6.23}
$$

at finite frequency, the Klein–Gordon equation for the massless scalar field ϕ reads

$$
f^2\Psi''+ff'\Psi'+\left[\omega^2-f\left(\frac{L(L+D-3)}{r^2}+f'\frac{D-2}{2r}+f\frac{(D-4)(D-2)}{4r^2}\right)\right]\Psi=0\,. \tag{6.24}
$$

Let's adopt again the dimensionless radial variable

$$x \equiv \left(\frac{r_s}{r}\right)^{D-3}. \tag{6.25}$$

The frequency-dependent equation (6.24) is identical to the one obtained above in the static limit, except for a term proportional to $\Psi(r(x))$ with coefficient

$$-\frac{4\omega^2 x^{-\frac{2}{D-3}}}{4(D-3)^2(x-1)x}. \tag{6.26}$$

To extract the result at linear order in ω, we are allowed to make a near-zone approximation and replace this term with [30, 146, 150, 151]

$$-\frac{4\omega^2 x^{-\frac{2}{D-3}}}{4(D-3)^2(x-1)x} \longrightarrow -\frac{4\omega^2 x^2}{4(D-3)^2(x-1)x}. \tag{6.27}$$

Note that this replacement does not affect the boundary condition at the horizon ($x = 1$), where the approximation becomes in fact exact. On the other hand, away from the horizon the approximation starts to be inaccurate at order ω^2. In other words, as long as we are interested in the response at linear order in ω this is good enough. After making the substitution (6.27), and doing the field redefinition

$$u(x) \equiv (1-x)^{\frac{i\omega r_s}{D-3}} x^{-\frac{D+2L-4}{2(D-3)}} \Psi(r(x)), \tag{6.28}$$

the approximated scalar equation takes the standard hypergeometric form (6.9) with parameters

$$a = \hat{L} + 1 - \frac{2i\omega}{D-3}, \qquad b = \hat{L} + 1, \qquad c = 2\hat{L} + 2, \qquad \text{with} \qquad \hat{L} \equiv \frac{L}{D-3}. \tag{6.29}$$

Following the procedure above, we can easily find the hypergeometric solution that satisfies the correct infalling boundary condition at the horizon—note that the solution for ϕ should go as $\phi \sim (r-r_s)^{-i\omega r_s/(D-3)}$ at the horizon, which means that $u(x)$ must be regular at $x = 1$. Expanding it at large distances, we can then easily extract the relative falloffs in r. The generalization of (6.17) including the linear correction in the frequency is

$$\begin{aligned} k_{\text{scalar}} + k^{(\omega)}_{\text{scalar}} &= \frac{\Gamma(-2\hat{L}-1)\Gamma(\hat{L}+1)\Gamma\left(1+\hat{L}-\frac{2i\omega r_s}{D-3}\right)}{\Gamma(-\hat{L})\Gamma(2\hat{L}+1)\Gamma\left(-\hat{L}-\frac{2i\omega r_s}{D-3}\right)} \\ &= \frac{\Gamma(-2\hat{L}-1)}{\Gamma(-\hat{L})^2}\frac{\Gamma(\hat{L}+1)^2}{\Gamma(2\hat{L}+1)} + \frac{2\pi\cot(\pi\hat{L})\Gamma(-2\hat{L}-1)\Gamma(\hat{L}+1)^2}{\Gamma(-\hat{L})^2\Gamma(2\hat{L}+1)}\frac{i\omega r_s}{D-3} + O(\omega^2), \end{aligned} \tag{6.30}$$

with

$$k^{(\omega)}_{\text{scalar}} = \frac{2\pi\cot(\pi\hat{L})\Gamma(-2\hat{L}-1)\Gamma(\hat{L}+1)^2}{\Gamma(-\hat{L})^2\Gamma(2\hat{L}+1)}\frac{i\omega r_s}{D-3} + O(\omega^2). \tag{6.31}$$

We stress that (6.30) can be trusted only up to linear order in ω.

6.2 Linear response fields: EFT calculation and matching

In the previous section, we obtained the full solutions (6.17), (6.22) and (6.30). In this section, we will perform a similar calculation in the worldline EFT. Again, we will focus on non-rotating objects and scalar perturbation. The logic straightforwardly extends to gravitational perturbations, which we will comment on at the end.

Let us start from the EFT (5.75), where $S_{\rm int}$ is given in (5.102),

$$S_{\rm int} \supset \int {\rm d}\tau\, e\, Q^{i_1\cdots i_L}(X)\, C^{(s)}_{i_1\cdots i_L}\,, \tag{6.32}$$

where we take $s=0$ and

$$C^{(0)}_{i_1\cdots i_L} = \partial_{(i_1}\cdots\partial_{i_L)_T}\phi\,, \tag{6.33}$$

for a scalar field ϕ. The Green's functions of the fields ϕ_1 and ϕ_2, defined on the two branches of the closed-time path contour, are given by

$$G_{IJ}(\tau,\tau') = \begin{pmatrix} \langle T\phi_1(\tau)\phi_1(\tau')\rangle & \langle\phi_2(\tau')\phi_1(\tau)\rangle \\ \langle\phi_2(\tau)\phi_1(\tau')\rangle & \langle\bar{T}\phi_2(\tau)\phi_2(\tau')\rangle \end{pmatrix}\,, \tag{6.34}$$

where $I=1,2$, and where T and $\bar{T}$ denote time ordering and reversed time ordering, respectively. Note that G_{11} is the standard Feynman propagator, while G_{12} is the Wightman function. Let us rotate the fields on the contour to the Keldysh basis [152] as follows:

$$\begin{pmatrix}\phi_-\\ \phi_+\end{pmatrix} \equiv \begin{pmatrix} 1 & -1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1-\phi_2\\ \frac{1}{2}(\phi_1+\phi_2)\end{pmatrix}, \tag{6.35}$$

where ϕ_- and ϕ_+ are sometimes referred to as "quantum" and "classical". Introducing the matrix $\mathfrak{R}$ of the change of basis,

$$\mathfrak{R} \equiv \begin{pmatrix} \frac{1}{2} & 1\\ -\frac{1}{2} & 1\end{pmatrix}, \qquad \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \mathfrak{R}\cdot\begin{pmatrix}\phi_-\\ \phi_+\end{pmatrix} \tag{6.36}$$

one finds the two-point functions in the Keldysh basis [153–157]:

$$G^K_{AB} = \mathfrak{R}^{-1}\cdot G\cdot(\mathfrak{R}^t)^{-1} = \begin{pmatrix} G_{--} & G_{-+}\\ G_{+-} & G_{++}\end{pmatrix} = \begin{pmatrix} 0 & iG_A\\ iG_R & G_H\end{pmatrix}\,, \qquad \phi_A = \begin{pmatrix}\phi_-\\ \phi_+\end{pmatrix}, \tag{6.37}$$

where G_A, G_R and G_H are the advanced, retarded and Hadamard Green's functions, respectively, and where $\mathfrak{R}^t$ denotes the transpose of $\mathfrak{R}$. The retarded propagator G_R can be read off from eq. (5.105),

$$G_R(\tau-\tau') = -i\langle[\phi(\tau),\phi(\tau')]\rangle\Theta(\tau-\tau')\,, \tag{6.38}$$

while

$$G_A(\tau-\tau') = -i\langle[\phi(\tau'),\phi(\tau)]\rangle\Theta(\tau'-\tau)\,, \tag{6.39}$$

$$G_H(\tau-\tau') = \frac{1}{2}\langle\{\phi(\tau),\phi(\tau')\}\rangle. \tag{6.40}$$

Note that the Keldysh basis indices A, B are raised/lowered with the off-diagonal matrix[23]

$$c^{AB} = c_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.41}$$

The tidal response of the object can be obtained by computing the field's one-point function in the presence of an external background source. In other words, we will expand ϕ as

$$\phi = \bar{\phi} + \varphi, \tag{6.42}$$

where $\bar{\phi}$ denotes the tidal source and φ is the response.

We can now proceed by integrating out the internal degrees of freedom X. At leading order, this corresponds to solving for Q using linear-response theory, e.g.,

$$\langle Q^{i_1\cdots i_L}(\tau)\rangle = \int \mathrm{d}\tau' G_R^{i_1\cdots i_L|j_1\cdots j_{L'}}(\tau-\tau')C^{(s)}_{j_1\cdots j_{L'}}(\tau')\,, \tag{6.43}$$

where $C^{(s)}$ represents a classical external source and G_R is the retarded Green's function. Plugging the linear-response solution for Q back into (6.32), we get an interaction term in the in-in action of the form:

$$\Gamma^{\text{in-in}}_{\text{int}} = \int \mathrm{d}\tau \mathrm{d}\tau' G_{AB}^{i_1\cdots i_L|j_1\cdots j_{L'}}(\tau-\tau')C^{(s)A}_{i_1\cdots i_L}(\tau)C^{(s)B}_{j_1\cdots j_{L'}}(\tau')\,, \tag{6.44}$$

which we can then use to compute the one-point function

$$\langle \varphi_+(t,\vec{x})\rangle = \int \mathcal{D}\varphi_+\mathcal{D}\varphi_-\; \varphi_+(t,\vec{x})\; \mathrm{e}^{i\Gamma^{\text{in-in}}[\varphi]}, \tag{6.45}$$

in the presence of a background $\bar{\phi}_+ \equiv \phi_+ - \varphi_+$. Note that we are computing the expectation value of $\varphi_+ \equiv \frac{1}{2}(\varphi_1+\varphi_2)$ because this is the field combination that has a classical interpretation in the Keldysh–Schwinger approach. For the external classical source we fix

$$\bar{\phi}_1 = \bar{\phi}_2 \equiv \bar{\phi}, \tag{6.46}$$

so that $\bar{\phi}_- = 0$ and $\bar{\phi}_+ = \bar{\phi}$. Using (6.44), we find [157]:[24]

$$\begin{aligned}\langle \varphi_+(t,\vec{x})\rangle &= i\int \mathrm{d}\tau_1\mathrm{d}\tau_2\, \langle \varphi_+(\vec{x},t)\partial_{(i_1}\cdots\partial_{i_L)_T}\varphi_-(\tau_2)\rangle G_{+-}^{(Q)i_1\cdots i_L|j_1\cdots j_{L'}}(\tau_2-\tau_1)\partial_{(j_1}\cdots\partial_{j_{L'})_T}\bar{\phi}_+(\tau_1)\\ &= -(-1)^L\int \mathrm{d}\tau_1\mathrm{d}\tau_2\,\partial_{(i_1}\cdots\partial_{i_L)_T}G_R^{(\varphi)}(\vec{x},t-\tau_2)G_R^{(Q)i_1\cdots i_L|j_1\cdots j_{L'}}(\tau_2-\tau_1)\partial_{(j_1}\cdots\partial_{j_{L'})_T}\bar{\phi}_+(\tau_1)\,.\end{aligned} \tag{6.47}$$

Note that what we are computing corresponds to the diagram in figure 7, with $t > \tau_2 > \tau_1$.

Let us now take for the external tidal source $\bar{\phi}_+(\tau_1)$ a time-dependent profile of the form

$$\bar{\phi}_+(\tau) = \mathrm{e}^{-i\omega\tau}\,\mathcal{E}_{i_1\cdots i_L}x^{i_1}\cdots x^{i_L}. \tag{6.48}$$

[23] Recall that the metric in field space in the basis (ϕ_1,ϕ_2) was $\mathfrak{g} \equiv \mathrm{diag}(1,-1)$. Thus, in the new basis, $c = \mathfrak{R}^t\cdot\mathfrak{g}\cdot\mathfrak{R}$.

[24] Recall that, in the Keldysh basis, indices are contracted using c in (6.41), e.g., $\phi^A = c^{AB}\phi_B = (\phi_+,\phi_-)$.

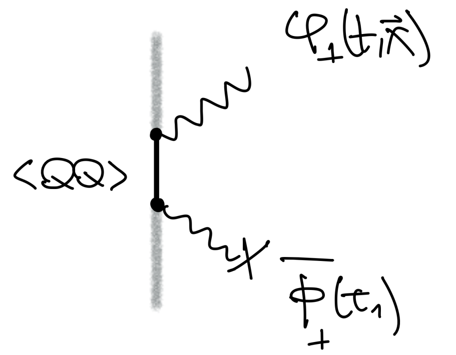

Figure 7: Feynman diagram corresponding to the induced response of a scalar field. Dissipation is parametrized by the two-point function of the composite operator Q.

Plugging this into the one-point function (6.47), and using the instantaneous (retarded) propagator for φ,[25] and the parametrization (5.96) with coefficients given by (5.97) in the non-rotating limit,

$$G_R^{i_1\cdots i_L|j_1\cdots j_L}(\omega) = [\lambda_0 + ir_s\omega\lambda_1 + \ldots]\, \delta^{(i_1}_{(j_1}\cdots\delta^{i_L)_T}_{j_L)_T}\,, \tag{6.49}$$

we get for the one-point function (6.47) [157]:

$$\begin{aligned}\langle\varphi_+(t,\vec{x})\rangle &= (-i)^L\int \mathrm{d}\tau_1\mathrm{d}\tau_2\int\frac{\mathrm{d}\tilde{\omega}}{2\pi}\frac{\mathrm{d}^d\vec{p}}{(2\pi)^d}\frac{p^{(i_1}\cdots p^{i_L)_T}}{\vec{p}^2}\delta(t-\tau_2)\,\mathrm{e}^{i\vec{p}\cdot\vec{x}}\,\mathrm{e}^{-i\tilde{\omega}(\tau_2-\tau_1)}\,(\lambda_0+ir_s\tilde{\omega}\lambda_1)\,\mathrm{e}^{-i\omega\tau_1}\,\mathcal{E}_{(i_1\cdots i_L)_T}\\ &= (-i)^L\,(\lambda_0+ir_s\omega\lambda_1)\,\mathcal{E}_{(i_1\cdots i_L)_T}\,\mathrm{e}^{-i\omega t}\int\frac{\mathrm{d}^d\vec{p}}{(2\pi)^d}\frac{p^{(i_1}\cdots p^{i_L)_T}}{\vec{p}^2}\,\mathrm{e}^{i\vec{p}\cdot\vec{x}}\,.\end{aligned} \tag{6.50}$$

Using the standard formulas

$$\int\frac{\mathrm{d}^d\vec{p}}{(2\pi)^d}\,\mathrm{e}^{i\vec{p}\cdot\vec{x}}\,\frac{1}{\vec{p}^2} = \frac{\Gamma(\frac{d}{2}-1)}{(4\pi)^{d/2}}\left(\frac{\vec{x}^2}{4}\right)^{1-\frac{d}{2}}\,, \tag{6.51}$$

$$i^L\int\frac{\mathrm{d}^d\vec{p}}{(2\pi)^d}\,\mathrm{e}^{i\vec{p}\cdot\vec{x}}\,\frac{p^{(i_1}\cdots p^{i_L)_T}}{\vec{p}^2} = \frac{\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{2^L(4\pi)^{d/2}\Gamma(2-\frac{d}{2}-L)}x^{(i_1}\cdots x^{i_L)_T}\left(\frac{x^2}{4}\right)^{1-\frac{d}{2}-L}\,, \tag{6.52}$$

we find, for the one-point function of the induced scalar field,

$$\langle\varphi_+(t,\vec{x})\rangle = (\lambda_0+ir_s\omega\lambda_1)\,\frac{(-1)^L\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{2^L(4\pi)^{d/2}\Gamma(2-\frac{d}{2}-L)}\,\mathrm{e}^{-i\omega t}\,\mathcal{E}_{i_1\cdots i_L}x^{i_1}\cdots x^{i_L}\left(\frac{\vec{x}^2}{4}\right)^{1-\frac{d}{2}-L}. \tag{6.53}$$

The full result for the scalar field, including the tidal source, is

$$\phi = \bar{\phi}+\langle\varphi_+\rangle = \mathrm{e}^{-i\omega t}\,\mathcal{E}_{i_1\cdots i_L}x^{i_1}\cdots x^{i_L}\left[1+(\lambda_0+ir_s\omega\lambda_1)\,(-1)^L\frac{2^{L+d-2}\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}\Gamma(2-\frac{d}{2}-L)}\frac{1}{r^{2L+D-3}}\right], \tag{6.54}$$

[25] Using $G_R^{(\varphi)}(\vec{p},t-\tau_2) = -\frac{1}{\vec{p}^2}\delta(t-\tau_2)$ is good enough for the response up to linear order in ω.

where $r \equiv \sqrt{\vec{x}^2}$ and $D = d+1$.

In the case of rotating objects, eq. (6.53) is straightforwardly generalized to

$$\langle\varphi_+(t,\vec{x})\rangle = \left(\lambda_0^{i_1\cdots i_L|j_1\cdots j_{L'}} + i r_s\omega\lambda_1^{i_1\cdots i_L|j_1\cdots j_{L'}}\right)\frac{(-1)^L\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{2^L(4\pi)^{d/2}\Gamma(2-\frac{d}{2}-L)}\,\mathrm{e}^{-i\omega t}\,\mathcal{E}_{j_1\cdots j_{L'}}x_{i_1}\cdots x_{i_L}\left(\frac{\vec{x}^2}{4}\right)^{1-\frac{d}{2}-L}, \tag{6.55}$$

which is valid up to linear order in ω.

Matching. The result (6.54) should be compared with the analogous calculation in the full theory. Let's perform the matching between (6.30) and the EFT result (6.54) for $\omega = 0$. We find, for the conservative static sector,

$$\lambda_0 = k_{\text{scalar}}\frac{(4\pi)^{d/2}(-1)^L\Gamma(2-\frac{d}{2}-L)}{2^{L+d-2}\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}, \tag{6.56}$$

where k_{scalar} is given in (6.17), with $d = D-1$, while, for the dissipative response at linear order in the frequency, we find

$$\lambda_1 = \frac{k_{\text{scalar}}^{(\omega)}}{i r_s\omega}\frac{(4\pi)^{d/2}(-1)^L\Gamma(2-\frac{d}{2}-L)}{2^{L+d-2}\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}, \tag{6.57}$$

where $k_{\text{scalar}}^{(\omega)}$ can be read off from (6.31).

Summary and outlook. In the section above, we have explicitly performed the matching for the static Love number coupling λ_0 and the dissipative number λ_1, up to linear order in the small frequency expansion, in a simple toy scalar field model on Schwarzschild spacetime. In the example, ω was the frequency of an external time-dependent perturbation in the lab frame, and the black hole was assumed to be non-rotating. We have shown that, in $D = 4$, $\lambda_0 = 0$ while $\lambda_1 \neq 0$. This result can be easily extended to gravitational perturbations and Kerr black holes, as well as to higher orders in ω. In the case of rotating objects, the matching is more naturally done in the rest frame of the body; in such cases, as opposed to Schwarzschild black holes, one finds that a dissipative response can be induced by an external tidal field that is static in the lab frame, but that appears to be time dependent in the frame that co-rotates with the black hole [99, 157].

In summary:

- In $D = 4$ general relativity, the gravitational static Love numbers λ_0 are zero for both Schwarzschild [10, 97, 120, 132–135] and Kerr [99, 136–138] black holes. For tidal perturbations that are time-independent (in the lab frame), there is no dissipative response for Schwarzschild black holes, while the dissipative response is non-vanishing for Kerr black holes. In the latter case, dissipation is a direct consequence of the rotation and is associated to frame dragging effects [99, 136–138].

- For Kerr black holes, the naive calculation of the response coefficients in full general relativity is affected by an ambiguity in the source/response split. In [99, 136, 137], such ambiguity was solved by performing an analytic continuation in L: one first obtains the static solution under the assumption that L is a real, non-integer number; one then computes the response numbers and sets L to the desired integer value in the result. A similar ambiguity is present for quadrupolar perturbations of Reissner–Nordström black holes [141].

- The vanishing of the Love numbers in four spacetime dimensions has been known as an outstanding naturalness puzzle in gravity [84, 158]. Following 't Hooft's naturalness principle, in the absence of symmetries, one in general expects Wilson coefficients to be order-one numbers, in units of the EFT cutoff scale. In this sense, the vanishing of black hole Love numbers in $D = 4$ appears to be associated with a fine tuning in the EFT. There are currently two main different proposals of solution to this puzzle: the first one is based on ladder symmetries of black hole perturbations [31, 149] (see also [141, 159]), while the second one relies on symmetries of a particular near-zone approximation of the linearized dynamics [160, 161]. In the case of scalar perturbations of Schwarzschild black holes, a link between the two sets of symmetries has been explained in [149].

More on the vanishing of the static Love numbers in four spacetime dimensions. It is instructive to look more closely at the vanishing of the static Love numbers in $D = 4$ from an EFT perspective. For simplicity, we will focus again on scalar perturbations around Schwarzschild black holes.

From a UV point of view, the vanishing of k_{scalar} in (6.17) is a consequence of the fact that the static equation (6.6) is a degenerate hypergeometric equation and the solution (6.12) that is regular at the black hole horizon has no decaying falloff at infinity in $D = 4$, i.e., schematically,

$$\phi(r) \sim r^L + r^{L-1} + \cdots + 1, \tag{6.58}$$

with constant coefficients. With the same logic of the metric reconstruction of section 5.2, one can recover the full solution (6.58), order by order in Gm, from the worldline EFT. This amounts to computing the diagrams in figure 8. Let's show this explicitly for first diagram with single insertion on the worldline, which corresponds to the first subleading correction to the r^L tidal profile in (6.58).

The worldline EFT of the compact object is (see eqs. (5.20), (5.35) and (5.36) in $D = 4$):

$$S_{\text{EFT}} = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_{\text{Pl}}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) - m\int \mathrm{d}\tau\sqrt{-g_{\mu\nu}v^\mu v^\nu}\,, \tag{6.59}$$

where $v^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\tau$. Let's expand it around the Minkowski background and keep terms up to the next-to-leading order in $1/M_{\text{Pl}}$. In order for the graviton action to be canonically normalized, we will define the

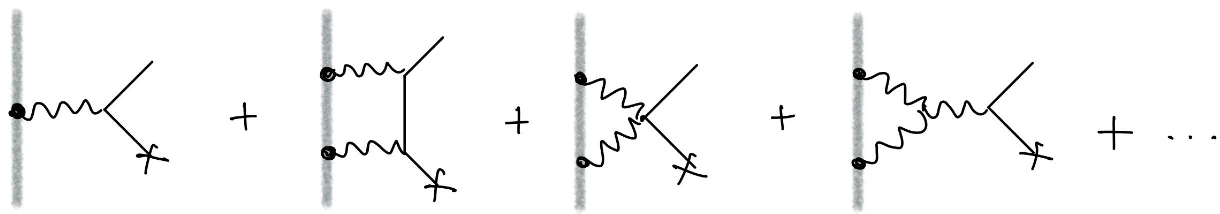

Figure 8: Reconstruction of the subleading falloff terms in the scalar solution (6.58). Wavy legs correspond to graviton propagators, while straight lines represent scalar legs. The dots refer to diagrams with higher number of mass insertions on the worldline, which are not shown explicitly.

metric perturbation as in eq. (5.38), i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_{\rm Pl}} h_{\mu\nu} \,, \tag{6.60}$$

where we are using the mostly-plus signature of the metric and $h_{\mu\nu}$ has dimensions of an energy. Up to linear order in $h_{\mu\nu}$, the worldline action (6.59) is

$$S_{\rm EFT} = \int {\rm d}^4 x \left[\frac{1}{2} h \mathcal{E} h - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \left(\eta^{\mu\alpha}\eta^{\nu\beta} - \frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu}\right)\frac{h_{\alpha\beta}}{M_{\rm Pl}}\partial_\mu\phi\partial_\nu\phi + \frac{m}{M_{\rm Pl}}\int {\rm d}\tau\, h_{\mu\nu}v^\mu v^\nu + \dots \right], \tag{6.61}$$

where $\mathcal{E}$ is the Lichnerowicz operator, and where we used the standard formulas

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} \,, \tag{6.62}$$

$$\delta g^{\mu\nu} = -g^{\mu\alpha}\delta g_{\alpha\beta} g^{\beta\nu} \,, \tag{6.63}$$

with $\delta g_{\mu\nu} = 2h_{\mu\nu}/M_{\rm Pl}$.[26] In the Lagrangian we will expand the scalar in background plus perturbations, $\phi = \bar\phi + \varphi$. For instance:

$$h_{\alpha\beta}\partial_\mu\phi\partial_\nu\phi = h_{\alpha\beta}\partial_\mu\bar\phi\partial_\nu\bar\phi + h_{\alpha\beta}\partial_\mu\bar\phi\partial_\nu\varphi + h_{\alpha\beta}\partial_\mu\varphi\partial_\nu\bar\phi + h_{\alpha\beta}\partial_\mu\varphi\partial_\nu\varphi \,. \tag{6.65}$$

From (6.61) and the split $\phi = \bar\phi + \varphi$, we can then read off the Feynman rules for the single-emission diagrams. The graviton and scalar propagators are

$$G^{\rm dD}_{\mu\nu\rho\sigma}(k) = \frac{i}{k^2}P_{\mu\nu\rho\sigma} = -\frac{i}{2k^2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right), \qquad G(k) = \frac{-i}{k^2}, \tag{6.66}$$

respectively (see also eq. (5.41)). The Feynman rule for the cubic vertex with one $\bar\phi$ and one φ is

$$-\frac{2i}{M_{\rm Pl}}\left(\eta_{\mu\alpha}\eta_{\nu\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\mu\nu}\right)k_2^\alpha k_3^\beta = -\frac{i}{M_{\rm Pl}}\left(k_{2\mu}k_{3\nu} + k_{2\nu}k_{3\mu} - \eta_{\mu\nu}k_{2\alpha}k_3^\alpha\right), \tag{6.67}$$

[26] By iterating (6.62) and (6.63), one can easily derive the expansion of $\sqrt{-g}$, e.g.,

$$\sqrt{-g} = \sqrt{-\eta}\Bigg[\frac{1}{M_{\rm Pl}}\eta^{\mu\nu}h_{\mu\nu} + \frac{1}{2M_{\rm Pl}^2}\left(\eta^{\mu\nu}\eta^{\alpha\beta} - 2\eta^{\mu\alpha}\eta^{\nu\beta}\right)h_{\mu\nu}h_{\alpha\beta} + \frac{1}{6M_{\rm Pl}^3}\left(\eta^{\mu\nu}\eta^{\alpha\beta}\eta^{\rho\sigma} - 6\eta^{\mu\rho}\eta^{\nu\sigma}\eta^{\alpha\beta} + 8\eta^{\mu\alpha}\eta^{\beta\rho}\eta^{\sigma\nu}\right)h_{\mu\nu}h_{\alpha\beta}h_{\rho\sigma} + O(h^4)\Bigg]. \tag{6.64}$$

while the mass insertion on worldline is

$$\frac{im}{M_{\rm Pl}}\int \mathrm{d}\tau\ \mathrm{e}^{-ik_\mu x^\mu(\tau)}\, v^\mu v^\nu . \tag{6.68}$$

Let's compute the diagram with single insertion of (6.67) (the first one in figure 8). In the absence of scalar charge, only the graviton line can be attached to the worldline. The resulting diagram is expected to reproduce the r_s correction to the external scalar field:

$$\begin{aligned}\langle\phi(x)\rangle = &-\frac{m}{2M_{\rm Pl}^2}\int \mathrm{d}\tau\, v^\mu v^\nu \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{e}^{-i(k-q)_\mu x^\mu(\tau)+ik_\mu x^\mu}}{k^2(k-q)^2}\bar\phi(q)\cdot\\ &\cdot\left(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma}\right)\left(-k^\rho q^\sigma-k^\sigma q^\rho+\eta^{\rho\sigma}k_\alpha q^\alpha\right)\\ &=\frac{m}{2M_{\rm Pl}^2}\int \mathrm{d}\tau\, v^\mu v^\nu \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{e}^{-i(k-q)_\mu x^\mu(\tau)+ik_\mu x^\mu}}{k^2(k-q)^2}\bar\phi(q)\left(k_\mu q_\nu+k_\nu q_\mu\right),\end{aligned} \tag{6.69}$$

which vanishes identically because of the form of the Fourier transform $\bar\phi(q)$ which is localized at $q=0$ (see, e.g., (6.48) with $\omega=0$). This is consistent with the full scalar solution in harmonic Schwarzschild coordinates (upon identifying m with the ADM mass M) [105]

$$\mathrm{d}s^2 = -\frac{r-GM}{r+GM}\mathrm{d}t^2+\frac{r+GM}{r-GM}\mathrm{d}r^2+(r+GM)^2\mathrm{d}\Omega_{S^2}^2, \tag{6.70}$$

which has in fact vanishing subleading correction in GM. The next-order correction is instead non-vanishing and can be recovered by computing the next Feynman diagrams in figure 8 with two mass insertions on the worldline. Proceeding in this way, one can eventually reconstruct the full scalar tidal field solution (6.58) [157].

Non-renormalization of static Love number couplings. In addition to reconstructing to full scalar solution from the EFT, one can also understand the absence of logarithmic running of the Love number couplings. On top of the bulk and point-particle action (6.59), let's add the quadratic, finite-size operators

$$S_{\rm int}^{\rm scalar} = \sum_L \frac{1}{L!}\int \mathrm{d}\tau\, e\,\lambda_L\left(\partial_{(i_1}\cdots\partial_{i_L)}\phi\right)^2, \tag{6.71}$$

in the conservative sector. Since we are interested in stationary fields, it is natural to perform a temporal Kaluza–Klein (KK) dimensional reduction for the metric [97, 115, 162]:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\,\mathrm{e}^{2\psi}\left(\mathrm{d}t-2\mathcal{A}_I\mathrm{d}x^I\right)^2+\mathrm{e}^{-\frac{2\psi}{D-3}}\gamma_{IJ}\mathrm{d}x^I\mathrm{d}x^J, \tag{6.72}$$

where ψ, $\mathcal{A}_I$ and γ_{IJ} correspond to the Newtonian potential, the gravito-magnetic vector and the three-dimensional metric, respectively, and where $I,J=1,\dots,D-1$. For time-independent fields, the Einstein–Hilbert action takes the form [115] (we will set $G=1$ for simplicity)

$$S_{\rm EH} = \frac{1}{16\pi}\int \mathrm{d}^Dx\sqrt{\gamma}\left(\mathcal{R}[\gamma]-\frac{D-2}{D-3}\partial_I\psi\partial^I\psi+\mathrm{e}^{2\frac{D-2}{D-3}\psi}F_{IJ}F^{IJ}\right), \tag{6.73}$$

where $F_{IJ}=\partial_I A_J-\partial_J A_I$ and $\mathcal{R}[\gamma]$ is the Ricci scalar of the metric γ. Here we are interested in the renormalization of the scalar couplings λ_L in (6.71). We have mentioned in section 5 that one can systematically reconstruct the background Schwarzschild metric by graviton insertions on the worldline. We shall take advantage of this fact by 'fixing a gauge' that is consistent with the GR metric reconstruction: in other words, we shall choose the fields in the KK metric (6.72) in such a way that the EFT calculation of the off-shell graviton one-point function automatically ensures the correct matching with the expanded background metric [97, 103, 157].[27] In practice, we will fix

$$\gamma_{IJ} = \delta_{IJ}(1+\sigma), \qquad \mathcal{A}_I = 0. \tag{6.74}$$

This choice matches the isotropic Schwarzschild coordinates [97]

$$\mathrm{d}s^2 = -\left(\frac{4r^{D-3} - r_s^{D-3}}{4r^{D-3} + r_s^{D-3}}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{r_s^{D-3}}{4r^{D-3}}\right)^{\frac{4}{D-3}} \left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{S^{D-2}}\right). \tag{6.75}$$

Then, the Einstein–Hilbert action for the perturbations schematically takes the following form:

$$S_{\mathrm{EH}} \sim \int \mathrm{d}^D x \left[(1+\sigma+\ldots)(\partial_I \psi)^2 + (1+\sigma+\ldots)(\partial_I \sigma)^2\right]. \tag{6.76}$$

Similarly, from the point-particle and scalar actions,

$$S_{\mathrm{pp}} \sim m \int \mathrm{d}\tau \left(1 + \psi + \psi^2 + \ldots\right), \tag{6.77}$$

$$S_{\mathrm{scalar}} \sim \int \mathrm{d}^D x \left(1 + \sigma + \sigma^2 \ldots\right)(\partial_I \phi)^2. \tag{6.78}$$

Note that, for static perturbations, the scalar ϕ couples only to σ and not to ψ. The opposite is true for the point particle: there are no σ insertions on the worldline. In addition, every vertex in the EFT involving σ always contains an even number of ψ fields. This implies that, at quadratic order in ϕ, it is only possible to draw classical worldline diagrams with an even number of ψ's, and hence an even number of point mass insertions on the worldline (we are disregarding bulk loops, which are subleading and do not contribute to the classical solution) [157]. This is consistent with the result of the calculation (6.69). In particular, it implies that, in $D = 4$, it is impossible to construct a diagram producing a $(2L+1)$PN-order correction to the one-point function, and that therefore the Love numbers do not get renormalized by graviton corrections [97, 157]. Note that this is no longer guaranteed in higher dimensions, when D is odd [10, 97].

Note that the generalization of this argument to diagrams with external graviton legs is not immediate. As opposed to ϕ, the odd fields $\mathcal{A}_I$ do couple directly to ψ, e.g. through a vertex $\sim \psi\mathcal{A}^2$ (see last term in (6.73)). It is thus a priori possible to draw a diagram with an odd number of mass insertions on the worldline and two external $\mathcal{A}_I$ fields. Hence, the previous power-counting rule is inconclusive for odd gravitational perturbations, at least with the chosen basis and gauge fixing (the absence of logarithmic corrections from power counting in the EFT is sufficient but not necessary).

[27] Alternatively, one can say that, instead of expanding the EFT around flat space, one expands the metric in the action around the perturbative Schwarzschild solution in suitable coordinates.

The aspects about self-force and radiation reaction, presented during the last lecture, can be mainly found in the reviews [163, 164]. For additional useful resources, see references therein, as well as e.g. [165–170], and the Capra meetings `webpage`.

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