Lorentzian methods in Conformal Field Theory

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1 Motivation

In recent years there was a huge progress in our understanding and applications of conformal fields theories in $d > 2$ dimensions, thanks to the revival of the "conformal bootstrap" [1]. Much of this work is in the Euclidean space, but also the Minkowski space makes its appearance and actually has a lot of applications. This raises the following two puzzles.

Q1. *Why is this useful?* 3d CFTs make predictions to second-order thermodynamic phase transitions in 3d systems (e.g. the 3d Ising model). Majority of these transitions are observed in Euclidean and at equlibrium (no time). Although one can imagine that one may formally analytically continue to Minkowski, naively statistical physicists should not care about this. However, the experience shows that it is sometimes very useful to have access to Minkowski, as some constraints on CFTs become much more visible there than in Euclidean. Partial list includes:

- Conformal collider physics [2]
- Lightcone analytic bootstrap [3][4]
- Constraints from causality and ANEC (averaged null energy condition) [5]
- Lorentzian OPE inversion formula [6] and its consequences (e.g. [7])

Q2. *Why is this permitted?* When phase transitions are observed on a Euclidean lattice, CFT limit is reached in the IR, at long distances. Short distances are dirty, contaminated with non universal lattice artifacts, and we don't think much about them. Yet when we go analytically to Minkowski, one the main questions is what correlation functions do at coincident points, are they distributions there? Why do we expect well-behaved short-distance singularities in Minkowski, given that on the lattice we have no control over coincident points?

In this course I will focus on Q2: under which conditions a Euclidean CFT gives rise to a good theory in Minkowski signature (and what we mean by a "good theory"). This is based on joint work with IAS postdoc Petr Kravchuk and my student Jiaxin Qiao, which should appear soon.

Note added (April 2021). *Results discussed in these lectures have since partly appeared in [8] and [9].*

2 Introduction to Euclidean CFT in $d > 2$

This section will cover standard material, see my EPFL lectures [10], or Simmons-Duffin's TASI lectures [11] for more details. For an extensive review of Euclidean CFT with an eye towards the bootstrap see [12]. Another highly recommended set of lectures, with exercises, is [13].

2.1 Conformal invariance

Main objects of study in CFT are correlation functions of local fields (a.k.a. operators)

$$\langle O_1(x_1)\, O_2(x_2) \ldots O_n(x_n)\rangle, \quad x_i \in \mathbb{R}^d. \tag{1}$$

One considers the group $\mathrm{Conf}(\mathbb{R}^d)$ of conformal transformations of $\mathbb{R}^d$ (more properly of $\mathbb{R}^d \cup \{\infty\}$). These are transformations $y = f(x)$ which locally look like a composition of a dilatation and a rotation. This means that the Jacobian matrix $J^\mu_{\ \nu} = \partial f^\mu / \partial x^\nu$ has the form

$$J^\mu_{\ \nu} = \Omega(x)\, R^\mu_{\ \nu}(x) \tag{2}$$

where $\Omega(x)$ is the scale factor, and $R(x)$ is an orthogonal matrix, $R\,R^T=1_{d\times d}$. Connected component of this group containing Identity is generated, in $d>2$, by translations+rotations, as well as dilatations $x\to\lambda\,x$ and SpecialConformalTransformations:

$$SCT(a)=I\circ T(a)\circ I \tag{3}$$

where $a\in\mathbb{R}^d$, $T(a)$ is the translation, and $I\colon x\to x^\mu/x^2$ is the inversion transformation. Inversion itself is a conformal transformation but it's not in the connected component ($R\in O(d)$ but not in $SO(d)$). Geometrically, one should think of SCT's as transformations which move ∞ while leaving 0 fixed (just like ordinary translations move 0 leaving ∞ fixed).

Exercise: Find Ω and R for inversion. Find the formula for how SCT act on x.

The algebra of conformal transformations is generated by P_μ, $M_{\mu\nu}$,D,K_μ (SCT generators) and is isomorphic to so$(d+1,1)$. More details will be given later if necessary.

CFT correlators are required to be invariant under conformal transformations, in the following sense. There is a linear transformation rule which acts on fields:

$$O(x)\to\pi_f[O](x),\quad f\in\mathrm{Conf}(\mathbb{R}^d) \tag{4}$$

so that

$$\langle O_1(x_1)\,O_2(x_2)\dots O_n(x_n)\rangle=\langle\pi_f[O_1](x_1)\,\pi_f[O_2](x_2)\dots\pi_f[O_n](x_n)\rangle \tag{5}$$

As we say "fields transform in a representation of conformal group". We are especially interested in representations such that $\pi_f[O](x)$ is proportional to $O(x')$ where $x'=f^{-1}(x)$:

$$\pi_f[O](x)\propto O(f^{-1}\,x) \tag{6}$$

Then Eq. (5) just says that correlation functions at two sets of points are proportional to each other. Note that such transformations have a chance to compose nicely so that $\pi_{fg}=\pi_f\,\pi_g$:

$$\begin{aligned}\pi_{fg}[O](x) &\propto O((f\,g)^{-1}\,x)=O(g^{-1}\,f^{-1}\,x),\\ \pi_f[\pi_g[O]](x) &\propto \pi_g[O]\,(f^{-1}\,x)\propto O(g^{-1}\,f^{-1}\,x).\end{aligned} \tag{7}$$

Prefactor in (6) can be fixed by demanding that the transformations indeed compose nicely (i.e. that we have a group representation). For scalar fields (i.e. fields without indices) we have a one parameter family of representations labeled by a number Δ:

$$\pi_f[O](x)=\Omega(x')^{-\Delta}\,O(x'),\quad x'=f^{-1}\,x. \tag{8}$$

where Ω is determined from f by (2). The meaning of parameter Δ can be understood by considering rigid dilatations $x\to\lambda\,x$ for which $\pi[O](x)=\lambda^{-\Delta}\,O(x/\lambda)$ and the invariance equation takes the form:

$$\langle O_1(x_1)\,O_2(x_2)\dots O_n(x_n)\rangle=\lambda^{-\Delta_1-\dots-\Delta_n}\,\langle O_1(x_1/\lambda)\,O_2(x_2/\lambda)\dots O_n(x_n/\lambda)\rangle. \tag{9}$$

It is then clear that Δ plays the role of a field's scaling dimension (each O will have its own Δ). In most physical situations we expect that Δ is a positive real number.

To generalize the transformation rule for fields with indices (vectors, tensors etc), recall that for rigid Poincaré transformations $x \to Rx + a$ we have

$$\pi_f[O](x) = \rho(R)\, O(x'), \tag{10}$$

where ρ is the matrix representing $SO(d)$ which acts on the indices of O. E.g. for vector representation we have $\rho(R) = R$, etc. Then, for conformal transformations this generalizes as:

$$\pi_f[O](x) = \Omega(x')^{-\Delta}\, \rho(R(x'))\, O(x'). \tag{11}$$

Exercise: Check that indeed $\pi_{fg} = \pi_f \pi_g$.

To summarize, in CFTs we are interested in fields O_i, infinite in number, each of which will be characterized by a scaling dimension Δ_i and an $SO(d)$ irrep ρ_i and whose correlation functions will be invariant under transformations (11). Such fields are called "primaries".

Note that derivatives of primary fields (called descendants) are not themselves primaries. E.g. transformations rules of $v_\mu = \partial_\mu O$ will have the schematic form

$$\pi_f\,[v] \sim v + O, \tag{12}$$

which is not homogeneous in v_μ. Under rather general conditions one can argue that any field in CFT is either or a primary or a derivative (of arbitrary order) of a primary. It is then sufficient to study correlators of primaries (however one cannot completely forget about descendants since they appear in OPE, see below).

In this course we will only discuss correlation functions of bosonic operators, such that they transform in tensorial representations of $SO(d)$. In fact we will mostly focus on scalars. Although not discussed here, there are interesting CFTs which contain fermionic operators transforming in spinorial representations, with the usual subtleties associated with double-cover nature of these irreps.

2.2 Constraints on correlation functions

Imposing constraints of conformal invariance in d dimensions, one arrives to interesting conclusions, as first done in 1970 by Polyakov [14]. Two point (2pt) functions are non-vanishing only among primaries transforming in the same representation (Δ, ρ), in which case its functional form is uniquely determined. By a change of basis we can assume that "2pt functions are diagonal", namely that each primary has two point function only with itself (there are exceptional cases called log-CFTs when this can't be done but we will ignore them here). We give here results for scalar and "spin-l" cases (symmetric traceless tensor of rank l):

$$\langle O(x)\, O(0)\rangle = \frac{\mathcal{N}}{|x|^{2\Delta}}, \tag{13}$$

$$\langle O_{\mu_1\ldots\mu_l}(x)\, O_{\nu_1\ldots\nu_l}(0)\rangle = \frac{\mathcal{N}}{|x|^{2\Delta}}\, T^{(2)}_{\mu_1\ldots\mu_l,\nu_1\ldots\nu_l}(x), \tag{14}$$

$$T^{(2)} = I_{\mu_1\nu_1}(x)\ldots I_{\mu_l\nu_l}(x) - \text{traces}, \tag{15}$$

$$I_{\mu\nu} = \delta_{\mu\nu} - \frac{2\, x_\mu x_\nu}{x^2}. \tag{16}$$

Conventionally the normalization factor is fixed $\mathcal{N} = 1$.

The tensor $T^{(2)}$ is an example of a "conformally invariant tensor structure". Going to 3pt functions, generically there are no restrictions on $\Delta_1, \Delta_2, \Delta_3$ and there are finitely many tensor structures whose number depends on $\rho_{1,2,3}$. Thus 3pt functions of primaries are fixed up to finitely many coefficients. In one important special case there is just one tensor structure, and thus one coefficient: scalar-scalar-spin l. For three scalars we have Polyakov's formula ($x_{ij} = x_i - x_j$):

$$\langle O_1(x_1)\, O_2(x_2)\, O_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{h_{123}}|x_{13}|^{h_{132}}|x_{23}|^{h_{231}}}, \quad h_{ijk} = \Delta_i + \Delta_j - \Delta_k. \tag{17}$$

with coefficient c_{123} whose normalization is physically significant once normalization of fields has been fixed via 2pt functions. For scalar(x_1)-scalar(x_2)-spin l (x_3) the formula includes an extra tensor structure factor

$$T^{(3)}_{\mu_1\ldots\mu_l} = V_{\mu_1}\ldots V_{\mu_l} - \text{traces}, \quad V_\mu = \frac{1}{|x_{12}||x_{13}||x_{23}|}\,(x_{13}^\mu x_{23}^2 - x_{23}^\mu x_{13}^2), \tag{18}$$

of total scaling dimension zero.

There is a nice way to derive these formulas called "embedding formalism", about which you can read in my EPFL lectures.

Starting from 4pt functions, such "pure kinematics" no longer fixes the functional form uniquely. E.g. for 4pt function of four identical scalars we have

$$\langle O(x_1)\, O(x_2)\, O(x_3)\, O(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}, \tag{19}$$

where $g(u,v)$ is a function of conformally invariant cross ratios

$$u = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2}, \quad v = u_{1\leftrightarrow 3}. \tag{20}$$

2.3 Operator product expansion (OPE) and crossing

Fortunately, the 4pt function $g(u,v)$ is not independent, but it can be expressed via the previously introduced "CFT data" Δ_i and C_{ijk} using the OPE.

OPE is written as

$$O_1 \times O_2 = \sum_k C_{12k}\, O_k \quad \text{(schematically)} \tag{21}$$

or in full form as

$$O_1(x)\, O_2(y) = \sum_k C_{12k}\, P\,(x-y, \partial_y)\, O_k(y), \tag{22}$$

where O_k are all infinitely many "exchanged primaries" with their OPE coefficients (same as the 3pt function coefficients in $\langle O_1\, O_2\, O_k\rangle$, see below), and $P\,(x-y, \partial_y)$ a differential operator (an infinite series in derivatives).

The OPE is used inside any n-point function $\langle O_1(x)\, O_2(y)\ldots\rangle$ to reduce it to a sum of $n-1$ point functions $\langle O_k(y)\ldots\rangle$ acted upon by P's. E.g. we have for 3pt functions:

$$\langle O_1(x)\, O_2(y)\, O_k(0)\rangle = C_{12k}\, P\,(x-y, \partial_y)\,\langle O_k(y)\, O_k(0)\rangle \tag{23}$$

In this case all the OPE terms in the r.h.s. except for the shown one vanish, since the 2pt function $\langle O_{k'}(y)\, O_k(z)\rangle$ vanishes for $k' \neq k$. The 3pt function $\langle O_1(x)\, O_2(y)\, O_k(0)\rangle$ in the l.h.s. is proportional to the same coefficient C_{12k} which cancels. Eq. (23) can thus be used to determine $P\,(x-y,\partial_y)$, at least in principle. It shows that P depends only on Δ's and ρ's of all fields, and on space dimension d.

For concreteness consider the scalar 3pt case. Expanding at small $z = x-y$, we have:

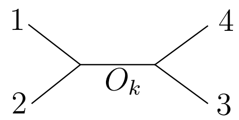
$$\begin{aligned}\langle O_1(y+z)\, O_2(y)\, O_k(0)\rangle &= \frac{1}{|z|^{\Delta_1+\Delta_2-\Delta_k}\,|y+z|^{\Delta_1+\Delta_k-\Delta_2}|y|^{\Delta_k+\Delta_2-\Delta_1}} \\ &= \frac{1}{|z|^{\Delta_1+\Delta_2-\Delta_k}|y|^{2\Delta_k}}\,(1+\text{series in } \frac{z}{y}) \qquad (24)\end{aligned}$$

From the leading term, we identify $\frac{1}{|y|^{2\Delta_k}}$ as the 2pt function $\langle O_k(y)\, O_k(0)\rangle$ in (23) and we require $P = \frac{1}{|z|^{\Delta_1+\Delta_2-\Delta_k}}\,(1+\text{series in } z\partial_y)$ to reproduce the subleading terms. Notice that the series will have to be divergent on the sphere $|z| = |y|$ because the 3pt function is singular when $x_1 = y+z \to 0$.

There are other ways to determine P. E.g. it can be fixed, in principle, by demanding correct conformal transformation properties. We won't need explicit expression for $P\,(x-y,\partial_y)$, but see EPFL lectures for an example (the first few terms), and Dolan and Osborn [15] for full expressions.

Using OPE twice, we can express the 4pt function as (four identical scalars $O = O_\Delta$ for concreteness)

$$\begin{aligned}\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle &= \sum_k C_k^2 P\,(x_{12},\partial_{x_2})P\,(x_{34},\partial_{x_4})\langle O_k(x_2)\, O_k(x_4)\rangle \\ &=: \sum_k C_k^2 G_{O_k}(x_i) \equiv \sum_k c_k^2 \qquad (25)\end{aligned}$$



where G_{O_k} is a function called conformal block fixed by conformal symmetry in terms of O_k's dimension and spin. Conformal blocks transform under conformal symmetry just as the 4pt function itself and should have the form

$$G_{O_k}(x_i) = \frac{g_{O_k}(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}} \qquad (26)$$

in terms of some function $g_{O_k}(u,v)$ (which is also sometimes called a conformal block). Computing conformal blocks for various 4pt functions is a big industry, see [12].

Conformal blocks in (24) are called "s-channel" because we apply OPE to points 12 and 34. S-channel expansion is expected to converge whenever there is a sphere which separates points 1,2 from points 3,4. We can also consider "t-channel" expansion when one applies OPE to points 23, 14. In overlapping region of convergence both expansions should agree. Such "crossing constraints" should be satisfied by any 4pt function.

2.4 Cutting and gluing picture

It's not our goal here to justify the usual rules of CFT. To make oneself comfortable, one may think of CFT correlators as arising from a path integral, which can be "cut and glued" along codimension 1 surfaces. We can fix boundary conditions $\phi|_S = \phi_0$ on a surface S and perform the path integral over all fields inside the surface (with insertions). The results of this computation, call it $\Psi_{\text{inside}}(\phi_0)$, is a wavefunctional, a "state" of fields living on S. We can also do the same computation over outside fields. When we finally take the product of two wavefunctionals and integrate over the boundary conditions:

$$\int D\phi_0 \Psi_{\text{outside}}(\phi_0) \Psi_{\text{inside}}(\phi_0) \tag{27}$$

this is an inner product of two states $(\Psi_{\text{outside}}, \Psi_{\text{inside}})$, and simultaneously the original correlator.

This discussion suggests that in any QFT we can think of any correlator as an inner product of two states living on any surface. Clearly there is a lot of freedom in choosing the surface, in particular for 4pt functions we can choose a surface separating 12 from 34 or 14 from 23, and this gives rise to a crossing constraint.

In a CFT, it's natural to use states living on spheres. Dilatation operator maps a sphere centered at the origin to another such sphere, and it's natural to choose a basis of its eigenstates. One can rationalize a lot of CFT lore using such mental pictures. See [16] for an attempt of rigorous exposition.

An extra comment is needed for OPE convergence. As the above intuitive argument shows, the space of CFT states in a sphere is naturally a vector space with a bilinear inner product. To argue for OPE convergence, we need to turn it to a Hilbert space (i.e. provide a positive-definite sesquilinear product). Basically, we need to provide a sesquilinear map $*$ which maps any inside-state Ψ to an outside-state Ψ^* such that their inner product $(\Psi^*, \Psi) \geqslant 0$. Such CFTs are called unitary and are an important subclass of CFTs. In Euclidean signature unitarity is also often called reflection positivity. Importantly, this condition is preserved under RG flow. E.g. if a lattice model is reflection positive, CFT describing its phase transition will be unitary.

Empirically, OPE converges also in some non-unitary CFTs, but in that case there is no robust argument known to me why this should be the case (except for 2d CFTs containing finitely many Virasoro primary fields).

2.5 Evidence for the validity of "bootstrap axioms"

We will call the above CFT rules (spectrum classification, OPE, crossing) "bootstrap axioms". Let's review evidence that these rules apply to actual theories describing physical phase transitions. First of all, we can consider any exactly solved 2d CFT with discrete spectrum, like minimal models. Of course 2d CFTs have a much larger symmetry (Virasoro) but let us forget about it and use only the global conformal group $SL_2(\mathbb{C})$. Then any Virasoro primary will give rise to infinitely many quasi-primaries=$SL_2(\mathbb{C})$ primaries. Virasoro conformal blocks will decompose as infinite sums of $SL_2(\mathbb{C})$ conformal blocks. So bootstrap axioms are satisfied in these examples.

In $d>2$, free massless theories (free scalar, free fermion, free Maxwell gauge field in 4d) satisfy bootstrap axioms, see [15] for some model computations.

Another class of solvable $d>2$ examples are gaussian theories based on non-local action called "Mean Field Theories". The simplest example is the scalar field with a non-local action $\int d^d x \phi (\partial^2)^s \phi$, which gives rise to the 2pt function $\langle \phi(x)\phi(0)\rangle = |x|^{-2\Delta}$ with $\Delta = d/2 - s$. Since the theory is gaussian, all higher-point functions are computed by Wick's theorem. Bootstrap axioms are satisfied and OPE coefficients can be exactly computed (see [17] for recent work).

Bootstrap axioms have been applied for CFTs corresponding to the main 3d Landau-Ginzburg universality classes (Ising, $O(N)$). Traditionally these CFTs are studied using perturbative expansion using the Lagrangian $(\partial\phi)^2 + \lambda(\phi^2)^2$ and going to the IR fixed point. Consider the Ising case for concreteness. The most relevant fields in the IR are the renormalized versions of ϕ and ϕ^2. The field ϕ^4, while relevant in the UV, has to become irrelevant in IR as the condition of reaching the fixed point. In the $\mathbb{Z}_2$-odd sector, the field ϕ^3 is expected to become a descendant of the primary ϕ thanks to the equation of motion $\phi^3 \sim \partial^2\phi$. The field ϕ^5 gets a large and positive anomalous dimension and will become even more irrelevant than ϕ^4. To summarize, the 3d Ising CFT should contain only two relevant scalar operators, one $\mathbb{Z}_2$-odd coming from ϕ, called σ, and one $\mathbb{Z}_2$-even coming from ϕ^2, called ε. Their OPEs should have the schematic form:

$$\begin{aligned} \sigma \times \sigma &= 1 + C_{\sigma\sigma\varepsilon}\varepsilon + \ldots \\ \sigma \times \varepsilon &= C_{\sigma\sigma\varepsilon}\sigma + \ldots \\ \varepsilon \times \varepsilon &= 1 + C_{\varepsilon\varepsilon\varepsilon}\varepsilon + \ldots \end{aligned} \tag{28}$$

where ... corresponds to all other primaries. As we said, other scalars should be irrelevant. We expect also primaries of spin $l>0$, in particular the stress tensor operator $T_{\mu\nu}$ of dimension 3. The ϕ^4 theory is unitary, and the 3d Ising CFT to which it flows should also be unitary. This implies lower bounds on operator dimensions of primaries called unitarity bounds (basically anomalous dimensions have to be positive), as well as reality constraints on OPE coefficients. Are there solutions to crossing satisfying all these constraints?

Very precise approximate solutions to crossing involving hundreds of exchanged operators have been constructed numerically, for the set of three 4pt functions $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\varepsilon\varepsilon\sigma\sigma\rangle$, $\langle\varepsilon\varepsilon\varepsilon\varepsilon\rangle$ [18]. These studies suggest that bootstrap axioms are consistent. These axioms turn out also very constraining: the set of allowed dimensions $\Delta_\sigma, \Delta_\varepsilon$ and OPE coefficients $C_{\sigma\sigma\varepsilon}, C_{\varepsilon\varepsilon\varepsilon}$ forms a tiny island. This gives [18]

$$(\Delta_\sigma, \Delta_\varepsilon, C_{\sigma\sigma\varepsilon}, C_{\varepsilon\varepsilon\varepsilon}) = (0.5181489(10), 1.412625(10), 1.0518537(41), 1.532435(19)).$$

Outside the island solution does not exist, while within the island it varies very little. This allows to predict CFT data of about a hundred primary operators of the 3d Ising CFT [19]. E.g., dimensions of the first irrelevant $\mathbb{Z}_2$-even and $\mathbb{Z}_2$-odd scalars are [19]

$$(\Delta_{\sigma'}, \Delta_{\varepsilon'}) = (5.2906(11), 3.82968(23)). \tag{29}$$

Summing the OPE, one can also compute and plot the 4pt functions, see [20] for $\langle\sigma\sigma\sigma\sigma\rangle$.

3 Other axiomatic schemes

Bootstrap being an axiomatic approach, it's interesting to see how it relates to other formerly proposed QFT axioms, notably Wightman and Osterwalder-Schrader axioms. Can we derive these axioms from the bootstrap?

3.1 Wightman axioms

Basic textbooks for Wightman axioms are [21],[22].

Let us leave CFT for a while and just discuss relativistic QFT. Generally accepted minimal properties of such theories are enshrined in Wightman axioms. One postulates the existence of a Hilbert space $\mathcal{H}$ on which the (unit-connected component of) Poincare group acts with unitary transformations U_g, $g=(a,\Lambda)$. Translations are represented by $e^{iP.a}$. One assumes that the spectrum of P^μ is in the closed forward light cone $\overline{V_+}$,[1] and that there is the unique invariant state Ω (vacuum): $U_g\Omega=\Omega$. Local operators $\varphi(x)$ are quantum-mechanical operators acting on the Hilbert space. They transform as

$$U_g^{-1}\varphi(x)U_g=\rho(\Lambda)\varphi(g^{-1}x) \tag{30}$$

where ρ is an irrep of Lorentz. Operators commute at spacelike-separated points (causality):

$$[\varphi(x),\varphi(y)]=0, \qquad (x-y)^2<0 \tag{31}$$

(we use the mostly minus metric). In general there will be many local operators, but here we limit for simplicity to just one $\varphi(x)$ which we assume hermitean: $\varphi^\dagger=\varphi$.

One then considers correlation functions of local operators:

$$\mathcal{W}_n(x_1,\ldots,x_n)=(\Omega,\varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\Omega) \tag{32}$$

These are called Wightman functions to distinguish them from time-ordered correlators (which won't be discussed here). By the above, they are Lorentz-invariant and commute at spacelike separation.

One would like to say something about their regularity, including at coincident points. An axiom says that Wightman functions should be tempered distributions,[2] i.e. the following integrals are finite:

$$\int dx_1\ldots dx_n\,\mathcal{W}_n(x_1,\ldots,x_n)f_1(x_1)\ldots f_n(x_n), \tag{33}$$

and depend continuously on test functions $f_i\in\mathcal{S}$ (Schwartz class). These can also be written as

$$(\Omega,\varphi(f_1)\varphi(f_2)\ldots\varphi(f_n)\Omega), \qquad \varphi(f)\equiv\int d^dx\,\varphi(x)f(x). \tag{34}$$

1. E.g. if the theory has particles of mass m we must have $P^2=m^2>0$.

2. We will abusively use both "Wightman functions" and "Wightman distributions".

where we defined smeared operators $\varphi(f)$. One then requires that an arbitrary string of smeared operators acting on Ω should produce a vector of finite norm. Without smearing this is false, e.g. the norm of $\varphi(x)\Omega$ is infinite: $(\varphi(x)\Omega, \varphi(x)\Omega) = (\Omega, \varphi(x)\varphi(x)\Omega) = \infty$. The need to smear is expressed by saying that "$\varphi(x)$ is an operator-valued distribution."

A few more important properties of Wightman distributions follow from the above definitions. Using translations, we can write $\mathcal{W}_{n+1}$ in the form

$$W_n(\xi_1, \ldots, \xi_n) = (\Omega, \varphi(0)e^{iP\cdot\xi_1}\varphi(0)e^{iP\cdot\xi_2}\ldots e^{iP\cdot\xi_n}\varphi(0)\Omega) \tag{35}$$

where $\xi_1 = x_2 - x_1$, $\xi_2 = x_3 - x_2$ etc. Since P has spectrum in forward light cone, we conclude that the Fourier transform of W_n (which exists since this is a tempered distribution) vanishes unless each momentum q_i is in the forward light cone (*spectral property*).

Another property follows from positivity of the norm. Consider a general finite linear combination of states created by jointly smeared operators:

$$\Psi = \sum_n \varphi(x_1)\varphi(x_2)\ldots\varphi(x_n) f_n(x_1, \ldots, x_n)\Omega \tag{36}$$

(the $n = 0$ term is just $f_0\Omega$). Expressing positivity of its norm $(\Psi, \Psi) \geqslant 0$ we get the *positivity property* of Wightman distributions:

$$(\Psi, \Psi) = \sum_{n,m} \int \mathcal{W}_{n+m} \cdot ((f_n^*)^{\text{inv}} \otimes f_m) \geqslant 0, \tag{37}$$

which has to be true for any finite sequence of test functions f_n.[3]

To summarize, Wightman axioms postulate Hilbert space with unitary representation of Lorentz group, a spectral conditon for P, and operator-valued distributions for causal local operators. Then one derives properties of Wightman distributions. One can also reverse the logic: start from Lorentz invariant causal Wightman functions (distributions) which satisfy the spectral and positivity properties. *Wightman reconstruction theorem* says that one can then recover the Hilbert space and other structures.[4] We can thus forget about the Hilbert space and just talk about Wightman distributions.

3.2 Analytic continuation of Wightman functions

Wightman functions allow analytic continuation to complex coordinates $z_k \in \mathbb{C}^d$ such that $\text{Im}(z_k - z_{k-1}) \in V_+$. Indeed consider the Fourier integral representation

$$W_n(\xi_1, \ldots, \xi_n) = \int \hat{W}_n(q_1, \ldots, q_n) e^{i\sum q_k\cdot\xi_k} \tag{38}$$

3. $()^{\text{inv}}$ means that the order of the arguments must be inverted; in full notation the generic term reads $\int dx_1 \ldots dx_{n+m} \mathcal{W}_n(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) f_n^*(x_n, \ldots, x_1) f_m(x_{n+1}, \ldots, x_{n+m})$.

4. The proof of this theorem is not hard. One knows that the theory should contain states (36) associated with any sequence of test functions. So one defines the Hilbert space to consist of states being sequences of test functions, and one defines the norm by (37). Then one just follows one's nose.

If each $\mathrm{Im}\,\xi_k \in V_+$ then $e^{i\sum q_k \cdot \xi_k}$ is a decreasing exponential in $\overline{V_+}$. Since the support of $\hat{W}$ is in $\overline{V_+}$. we can also stick some bump function equal to 1 inside the lightcone, which does not change the result. Then the integral looks like applying a distribution to a test function, thus finite and analytic in $\mathrm{Im}\,\xi_k \in V_+$.

We can say that Wightman functions W_n, which are distributions, are boundary values of functions analytic on the set $\mathrm{Im}(\xi_k) \in V_+$ ("forward tube"). This has many interesting consequences, of which we will explore only some.

We are mostly interested in analytic continuation to the Euclidean space. Euclidean time τ is related to the Lorentzian time t by $t = -i\tau$.[5] Euclidean correlators (*Schwinger functions*) are going to be functions of an unordered set of Euclidean coordinates $\{x_1^E, \ldots, x_n^E\}$. We will write it as $\mathcal{S}_n(x_1^E, \ldots, x_n^E)$ but keep in mind that the order of the arguments is unimportant, it's a function of an unordered set (unlike Wightman functions for which order of arguments was important unless points are spacelike separated).

For a given set $\{x_1^E, \ldots, x_n^E\}$ we **define** Schwinger function as follows. Pick some Euclidean time axis and reorder points so that their Euclidean times with respect to this axis are decreasing: $\tau_1 > \ldots > \tau_n$. Then compute the function by:

$$\mathcal{S}_n((\tau_1, \mathbf{x}_1), \ldots) = \mathcal{W}_n((-i\tau_1, \mathbf{x}_1), \ldots) \tag{39}$$

Since $\mathrm{Im}(z_k - z_{k-1}) = (\tau_{k-1} - \tau_k, \mathbf{0})$ is in the forward tube, this definition makes sense and defines a locally analytic function, which is invariant under Euclidean rotations, which are precisely complex Lorentz transformations preserving the set of points $(-i\tau, \mathbf{x})$.

So this is how Schwinger functions are computed, but we still have to convince ourselves that this definition is consistent. What if we choose another Euclidean time axis, will different analytic continuations agree? For small changes of the axis which do not change the Euclidean time order this is guaranteed by rotation invariance. For big changes of axis, as in Fig. 1, this is also true but a bit harder to see.

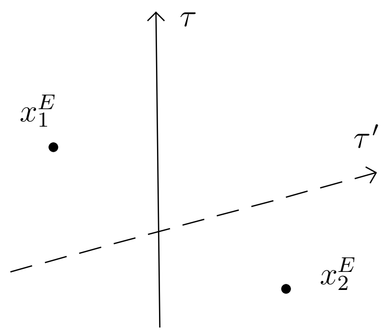


Figure 1. Schwinger functions do not depend on the choice of the Euclidean time axis used to define them, but this requires an argument, see the text.

For the sake of completeness we will give an argument. This will be a small detour but it will let us mention a few nice properties of Wightman functions.

It goes without saying that analytic continuations we are discussing remain Lorentz invariant:

$$W_n(\Lambda \xi) = W_n(\xi)\,. \tag{40}$$

5. This is easy to remember: Schrodinger evolution operator e^{-iHt} should become decaying exponential $e^{-H\tau}$ in Euclidean.

Moreover by analyticity this remains true also for complexified Lorentz transformations obtained by taking $\Lambda = \exp(i\alpha_i L_i)$, where L_i are the Lorents generators, and promoting α_i to complex numbers.[6] The image of the forward tube under complex Lorentz is called the 'extended tube', it's a strictly large open set.[7] We can then use Eq. (40) to extend Wightman functions to the extended tube, and Bargmann-Hall-Wightman theorem asserts that no ambiguities arise (any two continuation paths can be continuously deformed one into the other and thus give the same answer).

One interesting obsevation is that although the extended tube contains some points belonging to the original real Minkowski space $\mathbb{R}^{1,d}$ from where we started our analytic continuation. These points are called "Jost points" and they correspond to (some, not all) totally spacelike point configurations. See [22], p.81, for a characterization of Jost points. This is interesting because it shows that Wightman functions are analytic on some portion of Minkowski space corresponding to totally spacelike configurations.[8]

Notice furthermore that Wightman functions are symmetric at totally spacelike configurations because of causality. Thus they are symmetric at Jost points. Denote by Σ_n' point confugurations corresponding to point differences lying in the extended tube, and Σ_n^P point configurations corresponding to all possible permutations of Σ_n'. What the above shows is that Wightman functions can be analytically continued to Σ_n^P and are symmetric functions in this domain, invariant under complex Lorentz.

After this long preparation let us explain why the definition of Schwinger functions is consistent. Different choices of τ axis correspond to a different permutation of Wightman function arguments, but all permutations are in fact equal as we have shown. End of proof.

We have used the spectral property of Wightman functions to construct the Schwinger functions and prove their analyticity. Using positivity condition, we will now show that Schwinger functions satisfy a property called reflection positivity. As a simple but rather representative case, consider the 4pt Wightman function which we will write as

$$(\Omega, \varphi(0, \mathbf{x}_1) e^{-iH(t_1-t_2)} \varphi(0, \mathbf{x}_2) e^{-iH(t_2-t_3)} \varphi(0, \mathbf{x}_3) e^{-iH(t_3-t_4)} \varphi(0, \mathbf{x}_4) \Omega) \tag{41}$$

6. Equivalently complex Lorentz transformations can be defined as complex matrices solving the same equation $\Lambda^T \eta \Lambda = \eta$ as real Lorentz transformations.

7. Consider 2d example. In lightcone coordinates $x^{\pm} = t \pm x$, Lorentz acts by $x^+ \to x^+ e^{\lambda}$, $x^- \to x^- e^{-\lambda}$, $\lambda \in \mathbb{R}$, complex Lorentz by the same equations with $\lambda \in \mathbb{C}$, forward tube condition is $\operatorname{Im} x^{\pm} > 0$, i.e. both of them in the upperhalfplane. By choosing $\lambda = i\theta$, $x^{\pm}$ are rotated in the opposite directions, and if θ is large enough they will get out of the upper halfplane.

8. One can also show that $\mathcal{W}_n$ are analytic at **any** confugration of points which is totally spacelike (all pairwise separations spacelike), without using Jost points. For such configurations we can reorder points at will, e.g. reverse the order:

$$\mathcal{W}_n(x_1, \ldots., x_n) = \mathcal{W}_n(x_n, \ldots., x_1).$$

If we pass to point differences, then the l.h.s. gives $W_n(\xi_1, \ldots, \xi_n)$ while the r.h.s. $W_n(-\xi_n, \ldots, -\xi_1)$. Both functions can be analytically continued, the first one to $\operatorname{Im} \xi_k \in V_+$, the second to $\operatorname{Im}(-\xi_k) \in V_+$. So $W_n(\xi_1, \ldots, \xi_n)$ allows analytic continuations to both forward and backward null cone. By the famous edge-of-the-wedge theorem ([21], Theorem 2-16), $W_n(\xi_1, \ldots, \xi_n)$ then allows analytic continuation to an open neighborhood, i.e. is analytic at such points.

where we put all operators to zero time but did not care to shift the spatial coordinates. Continuing to Euclidean we get

$$(\Omega, \varphi(0, \mathbf{x}_1) e^{-H(\tau_1-\tau_2)} \varphi(0, \mathbf{x}_2) e^{-H(\tau_2-\tau_3)} \varphi(0, \mathbf{x}_3) e^{-H(\tau_3-\tau_4)} \varphi(0, \mathbf{x}_4) \Omega) \tag{42}$$

Now let us suppose that the spatial coordinates are pairwise identical $\mathbf{x}_4 = \mathbf{x}_1$, $\mathbf{x}_3 = \mathbf{x}_2$, while time coordinates are $\tau_1 > \tau_2 > 0 > \tau_3 > \tau_4$ are symmetric w.r.t. $\tau = 0$: $\tau_3 = -\tau_2$, $\tau_4 = -\tau_1$. As we say this point configuration is "reflection symmetric" with respect to the plane $\tau = 0$. Then the correlator becomes

$$(\Omega, \varphi(0, \mathbf{x}_1) e^{-H(\tau_1-\tau_2)} \varphi(0, \mathbf{x}_2) e^{-H\tau_2} e^{-H\tau_2} \varphi(0, \mathbf{x}_2) e^{-H(\tau_1-\tau_2)} \varphi(0, \mathbf{x}_1) \Omega) = (\Psi, \Psi) \geqslant 0,$$

where

$$\Psi = e^{-H\tau_2} \varphi(0, \mathbf{x}_2) e^{-H(\tau_1-\tau_2)} \varphi(0, \mathbf{x}_1) \Omega. \tag{43}$$

Thus Schwinger functions are positive in reflection-symmetric configurations. This is a partial case of "reflection positivity", the full case formulated as follows. First, we can integrate it with respect to a symmetric weight function. Second, we can mix correlators of different order, just as in the positivity condition. Let $f_n(x_1, \ldots, x_n)$ be a finite sequence of functions in the Euclidean space supported at positive Euclidean times: $\tau_i > 0$. Denote by Θ the reflection operation: Θ: $(\tau, \mathbf{x}) \mapsto (-\tau, \mathbf{x})$, and acting on functions by

$$\Theta f_n(x_1, \ldots, x_n) = f_n(\Theta x_n, \ldots, \Theta x_1), \tag{44}$$

reflecting arguments and reversing their order. Then we have (*reflection positivity*):

$$\sum_{n,m} \int \mathcal{S}_{n+m} \cdot ((\Theta f_n)^* \otimes f_m) \geqslant 0. \tag{45}$$

It is possible to show that *at coincident points Schwinger functions grow not faster than a powerlaw*. It's instructive to see how this comes about, and what fixes the power in the powerlaw. For two points approaching each other, let us choose Euclidean time axis so that the separation, τ, is along the time direction. Then the value of Schwinger function is obtained, in momentum space, by evaluating distribution $\hat{W}$ on the test function $f = e^{-q^0 \tau}$ which decreases exponentially in the lightcone and should be extended outside the lightcone to get a Schwartz-class functions. Now for any distribution $\mathcal{A}$ there is a seminorm of a certain finite order N

$$\| f \|_N = \max_x (1 + |x|^2)^{N/2} \max_{|\alpha| \leqslant N} |\partial^\alpha f(x)| \tag{46}$$

such that it's bounded by this seminorm: $|\int d^d x \mathcal{A} f| \leqslant C \| f \|_N$ (N and C depend on $\mathcal{A}$). Let us apply this to $\hat{W}$ and to $f = e^{-q^0 \tau}$. We only need to take the maximum over the lightcone: f can be extended out of the lightcone without increasing the seminorm appreciably. It's easy to see that this maximum grows as $\tau^{-\text{const.} N}$. This proves the above statement in italics.

This constructions allows to send one Euclidean point to another from any direction along the radius and understand how Schwinger functions behave in this limit. But it tells nothing what Schwinger functions do exactly at coincident points, how powerlike singularities are regulated.

This may remind you the philosophical discussion we had last time, about how from the lattice point of view asking about coincident-point behavior makes little sense.

One can try to *define* Schwinger functions at coincident points, making them distributions there. This would be a sort of regularization procedure. Whatever it is, such a definition would be something extra, not recoverable by analytic continuation from Wightman functions.

3.2.1 Relation to random distributions

(This subsection can be omitted on the first reading.) One approach to Euclidean QFT is via a theory of random distributions (see e.g. textbook [23]). One imagines a measure $d\mu$ on the space of distributions $\varphi \in \mathcal{S}'(\mathbb{R}^d)$, which allows to compute averages like

$$\int d\mu(\varphi) e^{\int \varphi(x) f(x) d^d x} \tag{47}$$

where f is a test function. Then moments of this measure are Schwinger functions of the field φ, which are therefore, in this framework, distributions, including at coincident points.

It may therefore look that this point of view is more powerful then recovering Schwinger functions by analytic continuation. This is not quite so. Random distribution point of view usually applies to just one, fundamental, field of the theory. It's not a priori clear how other, composite, fields, like φ^2, fit in this framework.

On the other hand Wightman axioms apply to correlation functions of any local field in the theory, be that elementary or composite.

3.3 Osterwalder-Schrader theorem

We have seen above how starting with Wightman functions one construct Schwinger functions via analytic continuation and proves their main property: reflection positivity.

The famous *Osterwalder-Schrader reconstruction theorem*, says that one can go the other way: starting from reflection-positive Schwinger functions recover Wightman distributions satisfying the spectral and positivity conditions. Published in 1973 in Communications of Mathematical Physics [24], their first argument was found to contain an error. An updated formulation of the theorem was published in the second paper in 1975 [25], and it remains state of the art. To save the theorem, they introduced an extra technical assumption on Schwinger functions, called *growth condition*.[9] Basically, this condition puts a bound on the growth of n-point Schwinger-functions as a function of n. The growth is measured via the integrals: $\int d^d x \mathcal{S}_n f$ where f is a Schwartz functions of n variables vanishing at coincident points with all its derivatives (call this class $S_{\neq}$). We can measure the size of f via seminorms $\|f\|_N$ defined in (46). Then the growth condition reads:

$$\left| \int d^d x \mathcal{S}_n f \right| \leqslant \sigma_n \|f\|_{Cn}, \qquad f \in S_{\neq}. \tag{48}$$

9. For reasons unclear to me they called it "linear growth condition". If anything, it should have been called "factorial growth condition".

where $C > 0$ is an integer constant and constants σ_n should grow not faster than a power of factorial:

$$\sigma_n \leqslant \alpha (n!)^{\beta}. \tag{49}$$

Few physicists heard about the need to impose the growth condition. I myself was shocked to discover this a couple of years ago. The condition is pretty annoying thing in that it involves all n-point functions at once. Suppose I am interested in the 4pt function. It would be great to have a condition on the Euclidean 4pt function which will guarantee that it's going to be well behaved in Lorentzian as well. But the theorem does not give that. Instead, to get a single correlator in Lorentzian I have to know something about **all** n-point functions in Euclidean! I have no idea how to check this for a general CFT satisfying bootstrap axioms (see section 3.3.2 below for the gaussian case).

3.3.1 Some ideas about the proof

I'd like to give some idea about the proof and why growth condition is important.

The conditions of the theorem talk only about correlation functions, but as a first step one introduces the Hilbert space. This construction formalizes the sentence frequently used in physics that "Hilbert space on a plane is generated by all operator insertions at negative Euclidean times". What this means mathematically is that Hilbert space elements are **posited** as sequences of test functions $(f_n)_{n \geqslant 0}$ supported at negative times, and their inner products are defined by Eq. (45). One starts with finite sequences, and then completes the space. This allows one to introduce a time-translation generator (Hamiltonian H) as an operator acting on this Hilbert space. A simple argument shows that H is a positive operator, so that $T_\tau = e^{-\tau H}$ is a contraction semigroup: $\|T_\tau\| \leqslant 1$.[10]

Thus any Schwinger n-point function can be written as

$$(\Psi_{n_1}, e^{-H\tau} \Psi_{n_2}) \tag{50}$$

where we split the points into $n_1 + n_2 = n$ and τ is the time separation between the two groups of points. Then, by basic Hilbert space technology, we can analytically continue $\tau \to \tau + is$.

Then one has to deal with two problems. First, we need to know how the result grows when the Euclidean time separations are sent to zero while imaginary parts are held fixed. It's in this limit that Wightman distributions are recovered. Suppose we are analytically continuing 4pt functions and we would like to know what happens when the *first* Euclidean time separation is sent to zero, so we are dealing with

$$(\Psi_1, e^{-H(\tau + is)} \Psi_3) \tag{51}$$

OS estimate this by Cauchy-Schwarz, bounding by

$$(\Psi_1, e^{-H\tau} \Psi_1)^{1/2} (\Psi_3, e^{-H\tau} \Psi_3)^{1/2} \tag{52}$$

10. $|(\Psi, T_\tau \Psi)| \leqslant \|\Psi\| \, \|T_\tau \Psi\| = \|\Psi\| \, (\Psi, T_{2\tau} \Psi)^{1/2} \leqslant \|\Psi\|^{1+1/2+\cdots+1/2^{k-1}} (\Psi, T_{2^k \tau} \Psi)^{1/2^k} \to \|\Psi\|^2$ as $k \to \infty$ since $(\Psi, T_{2^k \tau} \Psi)$ should go to a constant at large k by clustering (a weaker assumption that it may grow at most polynomially in $2^k \tau$ would also suffice).

which is a Euclidean 2pt function times a Euclidean 6pt function. This is the basic reason why they need some assumption on 6pt functions, even if one is interested only in 4pt functions.

Second, we have analytically continued in every time separately, but we need to analytically continue in all of them jointly. In the above 4pt function example, this means that state Ψ_3 in (51) will need to be analytically continued. But then the 6pt function in the Cauchy-Schwarz bound (52) is at analytically continued values of coordinates. It needs to be also bounded, which brings in 8pt functions, etc.

This is as much as can be explained given the limited time. The factorial growth of the coefficients σ_n is such that all bounds from an infinite sequence of analytic continuations fit nicely together.

3.3.2 Check of growth condition for gaussian scalar field

For gaussian scalar (of any dimension Δ, not necessarily $d/2-1$) one can argue as follows. We have for n test functions on $\mathbb{R}^d$:

$$\mathcal{S}_n(f_1 \times f_2 \times \cdots \times f_n) = \text{sum of products of Wick contractions}, \tag{53}$$

each of which is of the form $(f_i G f_j)$ where G is the Euclidean Green's function. We can in this case dispense with the non-coincident point condition, bound each Wick contraction by $A\|f_i\|_r\|f_j\|_r$ where r depends on the dimension Δ, and get a bound on the l.h.s. of (53) of the form: $w_n A^n \prod \|f_k\|_r$ where $w_n = (n-1)!!$ is the number of terms in the r.h.s. The argument given in the appendix of [25] shows how to pass from operators smeared separately to operators smeared jointly, and derive (48) (for all $f \in S$).

4 Towards CFT Osterwalder-Schrader theorem

I will next describe a construction which, for CFTs, allows to prove that Lorentzian 4pt correlation functions satisfy Wightman axioms (joint work with Petr Kravchuk and Jiaxin Qiao). Importantly, we can prove this theorem for 4pt functions, without ever talking about higher-point functions.

4.1 What do we need to show

Consider Euclidean CFT $(n+1)$-pt function $\mathcal{S}_{n+1}(x_1, x_2, \ldots, x_{n+1})$, of scalars for simplicity. By translational invariance we have

$$\mathcal{S}_{n+1}(x_1, x_2, \ldots, x_n) = S_n(\xi_1, \xi_2, \ldots, \xi_n), \quad \xi_1 = x_2 - x_1, \xi_2 = x_3 - x_2 \quad \text{etc.} \tag{54}$$

We have $\xi_k = (\tau_k, \boldsymbol{\xi}_k)$. We consider time ordering $\tau_k > 0$. We need to analytically continue

$$\tau_k = \varepsilon_k + i t_k, \quad t_k \in \mathbb{R}. \tag{55}$$

and send $\varepsilon_k \to 0$ to recover Lorentzian Wightman functions $W_n((t_1, \boldsymbol{\xi}_1), \ldots)$. The key step is to prove the following bound on the analytic continuation (we rename arguments of S_n by x_n):

$$|S_n(x_1, x_2, \ldots, x_n)| \leqslant \text{const.} P(\varepsilon_k^{-1}, |\mathbf{x}_k|, |\tau_k|) \tag{56}$$

i.e. a bound which grows at most as a polynomial when we send $\varepsilon_k \to 0$ or any coordinate to ∞. From this single polynomial bound, we can then conclude:

- The limit $\varepsilon_k \to 0$ exists and is a tempered distribution
- More specifically we can write S_n as a Fourier-Laplace transform:

$$S_n(x_1, \ldots, x_n) = \int dq_1 \ldots dq_n \hat{W}(q_k) e^{-\sum q_k^0 \tau_k + i \sum \mathbf{q}_k \cdot \mathbf{x}_k} \tag{57}$$

  where $\hat{W}(q_k)$ is some distribution supported at $q_k^0 \geqslant 0$. The Wightman functions are inverse Fourier transforms of $\hat{W}(q_k)$
- In addition, from Euclidean invariance of S_n one easily shows that $\hat{W}(q_k)$, and in particular its support, must be invariant under Lorentz transformations. Since it is Lorentz-invariant and contained in $q_k^0 \geqslant 0$, it must in fact be contained in the forward light cone. Thus we recover the spectral condition.

It's magic that a polynomial bound implies all of this; we will explain below why.

4.2 2pt and 3pt function examples

But first let us apply this technology to CFT 2pt and 3pt functions. Since they are given by explicit expressions, analytic continuation is straightforward. For 2pt functions we have:

$$S(x) = \frac{1}{(x^2)^\Delta} = \frac{1}{(\tau^2 + \mathbf{x}^2)^\Delta} = \frac{1}{((\varepsilon + it)^2 + \mathbf{x}^2)^\Delta} \tag{58}$$

This is nonsingular at $\operatorname{Re} \tau = \varepsilon > 0$ and moreover satisfies the uniform bound

$$|S_2(x)| \leqslant \frac{1}{\varepsilon^{2\Delta}} \tag{59}$$

So we conclude without any extra work that the limit as $\varepsilon \to 0$ is a tempered distribution satisfying the spectral condition.[11]

Analogously for 3pt functions

$$S(x_1, x_2) = \frac{1}{(x_1^2)^a (x_2^2)^b ((x_1 + x_2)^2)^c} \leqslant \frac{1}{(\varepsilon_1^2)^a (\varepsilon_2^2)^b ((\varepsilon_1 + \varepsilon_2)^2)^c} \text{ (in abs. value)} \tag{60}$$

As this is polynomially bounded, we conclude that the joint limit $\varepsilon_k \to 0$ is a tempered distribution satisfying the spectral condition. For 2pt function, one could imagine verifying spectral condition by an explicit calculation, but for 3pt function this looks rather formidable.

11. Note that "shifting the contour" argument could only show that it's a distribution when integrated w.r.t. analytic test functions.

4.3 Vladimirov's theorem

That polynomial boundedness implies Fourier-Laplace representation and existence of the limit is called Vladimirov's theorem [26]. I'd like to explain why this magic theorem is true, in 1d case (generalization to d dimensions being straightforward). Consider therefore an analytic function $S(x+iy)$ defined for $x>0$, $y\in\mathbb{R}$ and bounded polynomially in all directions $y\to\pm\infty$, $x\to 0,\infty$. We can write this as

$$|S(x+iy)|\leqslant\left(\frac{1}{x^r}+x^r\right)P(y), \tag{61}$$

where $P(y)$ is a polynomial and $r>0$. We want to show that

(a) The limit

$$\lim_{x\to 0}S(x+iy) \tag{62}$$

exists as a tempered distribution in y.

(b) S can be written as a Fourier-Laplace transform:

$$S(x+iy)=\int d\xi g(\xi)e^{-(x+iy)\xi} \tag{63}$$

where $g(\xi)$ is a tempered distribution supported at $\xi\geqslant 0$.

1. Clearly (b) would imply (a), but the proof starts with first showing (a). So we pick a test function and study the integral

$$\int S(x+iy)f(y)dy=h(x)\,. \tag{64}$$

We need to show that this has a limit as $x\to 0$. This looks a bit magic, because estimating naively by absolute value one would conclude that the integral may blow up. It won't blow up only because there are some cancelations which are not captured by the naive estimate. Recall the Sochocki formula:

$$\lim_{\varepsilon\to 0^+}\frac{1}{x+i\varepsilon}=\mathrm{PV}\frac{1}{x}-i\pi\delta(x) \tag{65}$$

PV represents a kind of cancelations whose existence we need to exhibit in general.

Going back to (64),[12] the first key idea is that we can estimate not just h but any its derivative. Indeed, by Cauchy-Riemann equations, x-derivatives of $h(x)$ can be transformed into y derivatives acting on S which then can be integrated by parts to act on f. Using then the polynomial bound (61) we get an estimate of **any** derivative $h^{(j)}(x)$ by $1/x^r$ times a seminorm of f of order depending on j and on degree of P (to make the integral convergent):

$$|h^{(j)}(x)|\leqslant\frac{C}{x^r}\|f\|_N,\quad N=N(j,P) \tag{66}$$

12. We follow the proof in [21], Theorem 2-10.

This is still growing as $x \to 0$. Here comes the second key ide: since we have this bound on any derivative, we can strengthen it recursively using the Newton-Leibnitz formula:

$$h^{(j-1)}(x) = -\int_x^1 h^{(j)}(x) + h^{(j-1)}(1) \tag{67}$$

Every time we use this, we obtain the bound on $h^{(j-1)}$ with the order of singularity in x reduced by 1 w.r.t $h^{(j)}$. Starting this process with j sufficiently large we run it until we get rid of singularity as $x \to 0$ altogether. We stop the process once we get a bound on $h^{(1)}(x)$ of the form: O(1) times a seminorm of f.[13] Using this bound in

$$h(x) = -\int_x^1 h^{(1)}(x) + h(1) \tag{68}$$

we conclude that the limit of $h(x)$ as $x \to 0$ exists for every f and is a continuous linear functional of f, i.e. a tempered distribution.

2. We next show (b). First of all notice that for every $x > 0$ we can write

$$S(x + iy) = \int d\xi g_x(\xi) e^{-iy\xi} \tag{69}$$

where $g_x \in S'$ is the Fourier transform of $S(x + iy)$ with respect to y (which exists as a tempered distribution). Using Cauchy-Riemann equation $\frac{\partial S}{\partial y} = i\frac{\partial S}{\partial x}$ we conclude that

$$\frac{\partial g_x}{\partial x} = -\xi g_x \tag{70}$$

i.e. $g_x(\xi) = g(\xi) e^{-x\xi}$ where $g(\xi)$ is some x-independent distribution. Since g and g_x are related by an exponential factor, we can so far only claim that $g \in D'$ (i.e. applicable to compact support test functions). Consider the inverse of (69),

$$g_x(\xi) = g(\xi) e^{-x\xi} = \int dy\, S(x + iy) e^{iy\xi} \tag{71}$$

and integrate it against a compactly supported test function $\varphi(\xi)$, we get:

$$\int d\xi\, g(\xi) e^{-x\xi} \varphi(\xi) = \int dy\, S(x + iy) \hat{\varphi}(y) \tag{72}$$

As $x \to 0$, the l.h.s. tends to (g, φ). Using the result proved in part 1, the r.h.s. tends to $\int dy\, S(iy) \hat{\varphi}(y)$ which exists in the sense of tempered distributions and so is bounded by some Schwartz-space seminorm $\|\hat{\varphi}\|_N$. We get

$$|(g, \varphi)| \leqslant \text{const.} \|\hat{\varphi}\|_N \leqslant \text{const.} \|\varphi\|_{N'} \tag{73}$$

where in the second inequality we used that Fourier transform is continuous in Schwartz space. This inequality, valid for any compactly supported φ, means that g extends to a tempered distribution on the whole Schwartz space.

13. An integrable in x bound on $h^{(1)}$ would also suffice.

Incidentally, we also proved that $\lim_{x\to 0} S(x+iy)$ is given by the FT of this tempered distribution g.

3. It remains to prove that g is supported at $\xi \geqslant 0$. Indeed by the above we have, for any $x > 0$, $g(\xi) = e^{\xi x} g_x(\xi)$ where $g_x(\xi)$ is the Fourier transform of $S(x+iy)$ in y. Now let us take the limit $x \to \infty$. (This is the only place where we will need **large** x.) In this limit $e^{\xi x}$ decreases exponentially for any strictly negative ξ, while $g_x(\xi)$ grows at most as a powerlaw because of a polynomial bound on $S(x+iy)$. So $g(\xi) = 0$ for $\xi < 0$.[14]

5 The 4pt function

Consider the Euclidean 4pt function, of 4 identical scalars. Preparing to analytically continuing, we write it as

$$\langle \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\, \varphi(x_4)\rangle = \frac{g(u,v)}{(x_{12}^2)^{\Delta}(x_{34}^2)^{\Delta}}, \tag{74}$$

where $g(u,v)$ is a function of conformally invariant cross ratios

$$u = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2}, \quad v = u_{1\leftrightarrow 3}. \tag{75}$$

If we know $g(u,v)$ we can analytically continue it by setting

$$x_k = (\varepsilon_k + i t_k, \mathbf{x}_k) \tag{76}$$

keeping to the region $\varepsilon_4 > \varepsilon_3 > \varepsilon_2 > \varepsilon_1$. We will first get bounds on this analytic continuation, and then use Vladimirov's theorem to show that 4pt Wightman functoon exists, is a tempered distribution, and satisfies spectral property.

It is customary in CFT to pass from u, v to coordinates $z, \bar{z}$ defined by:

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \tag{77}$$

Solving we find

$$z, \bar{z} = \frac{1}{2}\left(1 + u - v \pm \sqrt{(1+u-v)^2 - 4u}\right) \tag{78}$$

The meaning of these coordinates is better understood by fixing 3 points $x_1 \to 0$, $x_4 \to \infty$, $x_3 \to (1,0,0,0)$ and then rotating x_2 to lie in 12 plane: $x_2 \to (x, y, 0, 0)$. Then: $u = x^2 + y^2$, $v = (1-x)^2 + y^2$ and so

$$z, \bar{z} = x \pm i y. \tag{79}$$

Thus: in Euclidean signature $z, \bar{z}$ are complex and complex-conjugate of each other.

14. If unhappy with this intuitive reasoning, the argument may be made more 'rigorous' in its integrated version: show that g vanishes on test functions supported at $x < a < 0$.

The function $g(u, v)$ can be written as a power-series expansion in $z, \bar{z}$:

$$g(z, \bar{z}) = \sum_{h, \bar{h} \geqslant 0} p_{h, \bar{h}} z^h \bar{z}^{\bar{h}} \tag{80}$$

The first term in this expansion is simply 1, corresponding to the unit operator in the OPE. The other terms correspond to exchanges of states of scaling dimension $\Delta = h + \bar{h}$ and spin $s = h - \bar{h}$ under rotation in the 12 plane. In a unitary theory, coefficients $p_{h, \bar{h}}$ are positive numbers, being squares of real OPE coefficients.

As a first try, we would like to use this formula to get an analytic continuation of g. For simplicity, suppose we just analytically continue in

$$x_2 \to (x, y + it, 0, 0) \tag{81}$$

(we would like the 2nd coordinate to become "time" so that the other three points remain spacelike separated - simplest case). Now

$$z, \bar{z} = x \pm i\,(y + it) \to x \mp t \quad (y \to 0) \tag{82}$$

So we see that in this Lorentzian limit $z, \bar{z}$ become two independent real numbers. We can just substitute it into $g(z, \bar{z})$ and try to define the analytic continuation by

$$g \text{ at } (x, t) = g(z, \bar{z})|_{z, \bar{z} = x \mp t}. \tag{83}$$

(The analytic continuation of the prefactor in 4pt function is understood). Will this work? Yes but not everywhere. Indeed we need to show that the resulting series converges, to claim analyticity. So we need some bound. From positivity of coefficients $p_{h, \bar{h}}$ we have

$$|g(z, \bar{z})| \leqslant g(|z|, |\bar{z}|) \leqslant g(r, r), \quad r = \max\,(|z|, |\bar{z}|) = \max|x \pm t|. \tag{84}$$

The quantity $g(r, r)$ is the 4pt function series in the configuration $(0, r, 1, \infty)$, it will be finite for $r < 1$ and powerlaw bounded as $r \to 1$ by the OPE in the crossed channel. The bottomline is that this construction achieves analytic continuation of the 4pt function to the Lorentzian diamond $|x \pm t| < 1$, which is only a part of the full Lorentzian space.

Notice that the function g is analytic in the square. The prefactor $1/(x_{12}^2)^{\Delta}$ has a singularity when $x_{12}^2 = x^2 + (y + it)^2 = 0$ which for $y \to 0$ happens on the lightcones $x = \pm t$. The resolution of this singularity depends whether the limit is taken from positive or negative y. These limits correspond to the two orderings in the Lorentzian Wightman functions

$$(\Omega, \varphi(x_2)\varphi(x_1)\varphi(x_3)\varphi(x_4)\Omega) \quad \text{and} \quad (\Omega, \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\Omega)$$

Notice that only the mutual ordering of $\varphi(x_1)$ and $\varphi(x_2)$ is important because other operators are spacelike-separated.

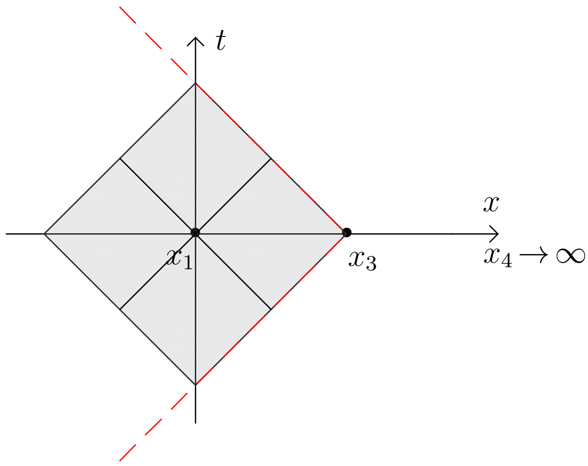


Figure 2. The described constructions based on the z coordinate allows to analytically continue the 4pt function in x_2 coordinate to the shaded "diamond" $|x \pm t| < 1$ in the Lorentzian plane (t, x). Using the ρ coordinate we will be able to extend this region to the larger "wedge" $x \pm t < 1$ (on left of the red dashed line)

By taking instead OPE in $x_2 - x_3$ channel, we could have constructed analytic continuation to a diamond centered on x_3. None of the two methods allows to control the Lorentzian point $x_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ which lies on both lightcones of x_1 and x_3. This "double lightcone singularity" is a mystery; there are few if any general results about what happens there.

5.1 The ρ coordinate

In addition to the more familiar $z, \bar{z}$, a smart coordinate to use in d-dimensional CFT is the ρ-coordinate [27],[28] related to $z, \bar{z}$ by:

$$\rho = \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \quad \text{or} \quad z = \frac{4\rho}{(1 + \rho)^2} \tag{85}$$

and analogously for $\bar{z}, \bar{\rho}$. Under this map, the plane of complex $z \in \mathbb{C} \backslash [1, +\infty)$ is mapped to the unit disk $|\rho| < 1$. There is a conformal transformation which maps the 4 points $(0, z, 1, \infty)$ to 4 points $(\rho, -\rho, 1, -1)$. The function g in ρ-coordinates has an expansion:

$$g(\rho, \bar{\rho}) = \sum_{h, \bar{h} \geqslant 0} q_{h, \bar{h}}\, \rho^h \bar{\rho}^{\bar{h}} \tag{86}$$

where q's are some positive coefficients. Thus ρ shares many useful properties of z, while being more powerful.

Let us try to use the ρ coordinate for the task of analytic continuation to Lorentzian. We consider the same continuation problem as in (81). We compute $z, \bar{z}$ by (82), then $\rho, \bar{\rho}$, and then g. If $|\rho|, |\bar{\rho}| < 1$, then we win big time: continuation will be analytic. As we just said, this will be true as long as $z, \bar{z}$ don't touch the cut $[1, +\infty)$ in the process of continuation. This in turn will be true if

$$\max\,(x \mp t) < 1, \tag{87}$$

which defines a "wedge", a larger region than the "diamond" $|x\pm t|<1$. Using the ρ coordinate, we extended the region of analyticity, but we still have no access to the double lightcone singularity and beyond.

But notice this: the whole of the region which we don't access ends up on the cuts: z or $\bar z$ or both are in $[1,+\infty)$, which means that ρ or $\bar\rho$ or both are on the circle $|\rho|,|\bar\rho|=1$. We are not analytic there, but we are just on the border of the region of analyticity (this was not at all obvious when using the z coordinate). This strongly suggests that 4pt function is a distribution on those regions, and to show this we just need to establish some powerlaw bounds on the growth when we approach the boundary of analyticity. In fact since $\rho(z)$ is an algebraic function it's essentially a given that some powerlaw bound will emerge.

Here is a simple proof of powerlaw bound, based on the Schwarz-Pick theorem. This theorem says that holomorphic maps from upper halfplane H **into** a unit disk D to itself satisfy the inequality:[15]

$$\left|\frac{f(z_1)-f(z_2)}{1-f(z_1)^*f(z_2)}\right|\leqslant\left|\frac{z_1-z_2}{z_1^*-z_2}\right| \qquad (f\colon H\to D). \tag{88}$$

Taking the limit $z_1\to z_2$ we get an inequality:

$$\frac{1}{1-|f(z)|^2}\leqslant\frac{1}{2|f'(z)|\,\mathrm{Im}\,z}, \quad f\colon H\to D \tag{89}$$

(we have an equality for $f(z)=\frac{z-i}{z+i}$ which maps H **onto** D).

Now we apply this to $f=\rho$. Differentiating $z=\frac{4\rho}{(1+\rho)^2}$ we find $\rho'(z)=\frac{(1+\rho)^3}{4(1-\rho)}$. Thus by (89)

$$\frac{1}{1-|\rho|^2}\leqslant\frac{\mathrm{const}}{|1+\rho|^3y} \qquad (z=x+iy). \tag{90}$$

The factor $1/(\rho+1)^3$ blows up only as $z\to\infty$ where we have $\rho=-1+O(1/\sqrt{z})$ so we get a polynomial bound

$$\frac{1}{1-|\rho|}\leqslant\frac{\mathrm{const}}{y}(1+|z|^{3/2})\,. \tag{91}$$

See section 5.1.1 for two more proofs of this bound. Notice that the 4pt functions will be bounded by $\frac{\text{const.}}{(1-\max|\rho|,|\bar\rho|)^{4\Delta}}$.

The bottomline is that, by this argument, we constructed Wightman functions in the whole (t,x) plane of x_2 and proved that they are distributions (including at the problematic future and past double lightcone discontinuities, and beyond). Depending which of the two limits $y\to0^\pm$ we take, we get Wightman functions

$$(\Omega,\boldsymbol{\varphi(x_2)}\varphi(x_1)\varphi(x_3)\varphi(x_4)\Omega) \quad\text{and}\quad (\Omega,\varphi(x_1)\varphi(x_3)\boldsymbol{\varphi(x_2)}\varphi(x_4)\Omega).$$

This is progress, but not the full resolution of the problem. Here is a wish list:

- So far we are considering configurations when 3 our of 4 points are mutually spacelike and fixed.

15. This theorem is usually stated as $\left|\frac{f(z_1)-f(z_2)}{1-f(z_1)^*f(z_2)}\right|\leqslant\left|\frac{z_1-z_2}{1-z_1^*z_2}\right|$ for $f\colon D\to D$ or as $\left|\frac{f(z_1)-f(z_2)}{f(z_1)^*-f(z_2)}\right|\leqslant\left|\frac{z_1-z_2}{z_1^*-z_2}\right|$ for $f\colon H\to H$. However for us the version given, for maps from H to D will be most convenient.

- Even for 3 points spacelike, not all possible operator orderings were constructed, e.g. what about $(\Omega, \varphi(x_1)\boldsymbol{\varphi(x_2)}\varphi(x_3)\varphi(x_4)\Omega)$?
- So far we are continuing just in one coordinate x_2 and just in a very special kinematical configuration (when all points lie in a 2d plane). What if we vary all 4 points, and arbitrarily, is it still a distribution?

5.1.1 Powerlaw bound on the approach $|\rho| \to 1$

Here we collect two more proofs of a polynomial bound on $\frac{1}{1-|\rho|}$.

1. Proof by a computation. Write

$$\rho = \frac{1-\sqrt{\zeta}}{1+\sqrt{\zeta}}, \qquad \zeta = 1 - z = a + iy. \tag{92}$$

Then

$$\rho\rho^* = \frac{1 - 2\mathrm{Re}\sqrt{\zeta} + |\zeta|}{1 + 2\mathrm{Re}\sqrt{\zeta} + |\zeta|} = \frac{1-\varepsilon}{1+\varepsilon}, \quad \varepsilon = \frac{2\mathrm{Re}\sqrt{\zeta}}{1+|\zeta|} > 0 \tag{93}$$

So we need to powerlaw-bound ε from below (Note that $\frac{1}{1-|\rho|} = \frac{1+|\rho|}{1-|\rho|^2} \leqslant \frac{2}{1-|\rho|^2} = \frac{1}{\varepsilon} + 1$). A simple computation shows:

$$\mathrm{Re}\sqrt{\zeta} = \sqrt{R/2}, \qquad R = \sqrt{a^2 + y^2} + a. \tag{94}$$

Considering positive and negative a separately, we can bound R below as

$$\begin{aligned} a \geqslant 0: &\quad R \geqslant \sqrt{a^2 + y^2} \\ a \leqslant 0: &\quad R = \frac{y^2}{\sqrt{a^2 + y^2} - a} \geqslant \frac{y^2}{\sqrt{a^2 + y^2}} \end{aligned} \tag{95}$$

Combining the two bounds, we get:

$$\varepsilon^{-1} \leqslant \mathrm{const.}\big(1 + \sqrt{a^2 + y^2}\big)\left(\frac{1}{(a^2 + y^2)^{1/4}} + \frac{(a^2 + y^2)^{1/4}}{y}\right) \tag{96}$$

which is a powerlaw bound of the kind we need to use Vladimirov's theorem.

2. A more structural argument. Consider the upper halfplane of $z = x + iy$ and how it and its boundary are mapped under the ρ map into the unit disk (Fig. 3). The problematic points near which $|\rho| = 1$ are on the cut $[1, \infty)$. We divide the upper halfplane into two regions: a neighborhood of the point $z = \infty$ (white) and the rest (gray). In the gray region we can treat $x, y = O(1)$ and it's easy to convince oneself that one has a bound

$$1 - |\rho| > \mathrm{const.}\, y. \tag{97}$$

In particular this is true both over the gray portion of the cut away from the branch point $z = 1$, but also near this branch point where ρ has a square-root singularity. [Hint: cover the boundary by a finite number of neighborhoods with local coordinates ζ_i, of the form $z - z_i$, or $(z - z_i)^{1/2}$ near the branch point, so that $\rho(z) = \rho(z_i) + A_i(\zeta_i)$ in each neighborhood, where A is an invertible analytic function.]

In the white region we can write ρ as

$$\rho = \frac{1 - \sqrt{1-z}}{1 + \sqrt{1-z}} = -1 - \frac{2}{1 + \sqrt{1-z}} = -1 + A\left(\frac{i}{\sqrt{z}}\right). \tag{98}$$

where A is a one-to-one analytic function. The map $z \to -1 + \frac{i}{\sqrt{z}}$ maps the white region to a straight quadrant formed the blue horizontal and the red dashed vertical lines, and writing $z = re^{i\varphi}$ we have

$$\mathrm{Re}\,\frac{i}{\sqrt{z}} > \mathrm{const.}\frac{\varphi}{r^{1/2}} \sim \mathrm{const.}\frac{y/\sqrt{x^2+y^2}}{(x^2+y^2)^{1/4}} \tag{99}$$

We also have

$$1 - |\rho| \sim \mathrm{Re}\,\frac{i}{\sqrt{z}} \tag{100}$$

because A is a smooth invertible function which maps the red dashed line to the red circle (within the white region). So we have powerlaw bounds everywhere.

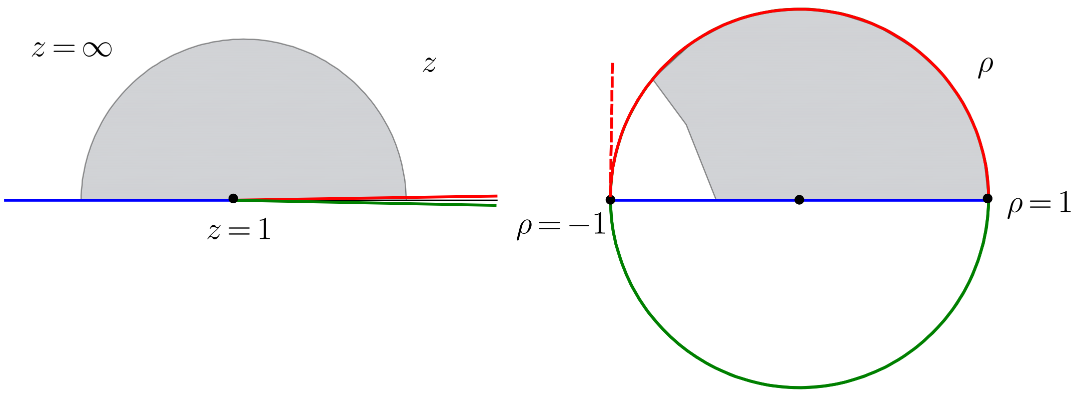


Figure 3. The map $\rho(z)$ from $\mathbb{C}\backslash[1,\infty)$ to the unit disk. 1 is mapped to 1, ∞ to -1, and curves to curves of the same color.

5.2 General 2d case

We will next consider the complete 2d case, i.e. all points will be allowed to move independently but they have to lie in a 2d plane $\mathbb{R}^2$ (which upon analytic continuation becomes Lorentzian $\mathbb{R}^{1,1}$). We will show that the 4pt function is a distribution. In fact for 4 points x_i arbitrarily positioned in the 2d plane and parametrized by the complex coordinates z_i, it's possible to compute the z and $\bar{z}$ parameters in closed form, as

$$z = \frac{z_1 - z_2}{z_1 - z_3}\frac{z_3 - z_4}{z_2 - z_4}, \tag{101}$$

and $\bar{z}$ analogously. We introduce $\zeta_i = z_i - z_{i+1}$ and rewrite it as

$$z = \frac{\zeta_1\zeta_3}{(\zeta_1 + \zeta_2)(\zeta_2 + \zeta_3)} = \frac{1}{(1 + \zeta_2/\zeta_1)(1 + \zeta_2/\zeta_3)} \tag{102}$$

We then analytically continue

$$\zeta_i = x_i + i y_i \to x_i + i\,(y_i + i t_i) = a_i + i y_i, \qquad a_i = x_i - t_i\,. \tag{103}$$

We then have to take the limit $y_i \to 0^+$ for completely arbitrary a_i. This will construct the Wightman function

$$(\Omega, \varphi(0,0)\varphi(x_1,t_1)\varphi(x_1+x_2,t_1+t_2)\varphi(x_1+x_2+x_3,t_1+t_2+t_3)\Omega), \tag{104}$$

in this operator ordering.

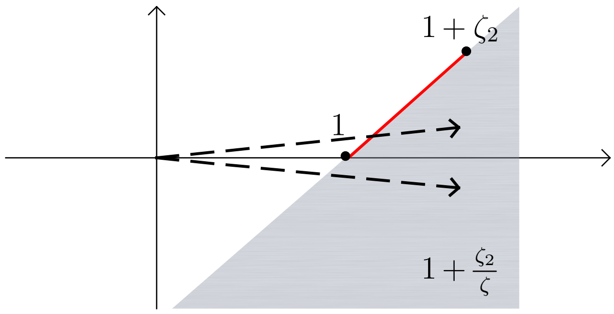


Figure 4. A proof that z does not cross the cut. Fix some $\zeta_2 \in H$, one example shown. Then $1+\frac{\zeta_2}{\zeta_1}$ and $1+\frac{\zeta_2}{\zeta_3}$ lie in the shaded region of the complex plane. To get z real, we have to pick them complex conjugate (e.g. two dashed vectors). But then their product is larger than 1, and so $z \in (0,1)$. We have shown that $z \in [1,\infty)$ is impossible.

The key observation is that as ζ_i vary over H, z does not cross the cut $[1,\infty)$. This is best shown graphically, see Fig. 4. So we can compute $\rho(z)$ and analytically continue the 4pt function as before. We need to show that as $\zeta_1, \zeta_2, \zeta_3$ vary over H^3, $|\rho(z(\zeta_1,\zeta_2,\zeta_3))|$ satisfies a polynomial bound in how fast it can approach 1.

The ρ is a function from H^3 to D. We can use the Schwarz-Pick theorem (89) with respect to any of the three coordinates, keeping the other two fixed. We thus have an inequality:

$$\frac{1}{1-|\rho|^2} \leqslant \frac{1}{2|\partial_{\zeta_1}\rho|y_1} \tag{105}$$

and similar ones for $\zeta_{2,3}$ (but this one will suffice). Furthermore

$$\partial_{\zeta_1}\rho = \partial_z\rho\, \partial_{\zeta_1} z \tag{106}$$

and using $\partial_z\rho$ given above we conclude:

$$\frac{1}{1-|\rho|^2} \leqslant \frac{\text{const.}}{|1+\rho|^3 |\partial_{\zeta_1} z|\, y_1} \leqslant \text{const}\, \frac{1+|z|^{3/2}}{|\partial_{\zeta_1} z|\, y_1}. \tag{107}$$

Now it's easy to see that both z and $\partial_{\zeta_1} z$ are polynomially bounded:

$$|z| = \frac{|\zeta_1\zeta_3|}{|\zeta_1+\zeta_2||\zeta_2+\zeta_3|} \leqslant \frac{|\zeta_1\zeta_3|}{|y_1+y_2||y_2+y_3|}, \tag{108}$$

$$|\partial_{\zeta_1} z| = \left|\frac{\zeta_1\zeta_3}{(\zeta_1+\zeta_2)(\zeta_2+\zeta_3)}\right| = \left|\frac{\zeta_2\zeta_3}{(\zeta_1+\zeta_2)^2(\zeta_2+\zeta_3)}\right| \leqslant \frac{|\zeta_2\zeta_3|}{|y_1+y_2|^2|y_2+y_3|}, \tag{109}$$

(in fact all $\partial_{\zeta_k} z$ are similarly bounded). End of proof.

5.3 General case

Note added (April 2021). *The main goal of this section is to prove inequality (121). This inequality is correct, but the proof given in this section does not quite work. The problem is that we have not justified that the series in the r.h.s. of (115) converges. For this reason in Ref. [9] we followed a different route towards showing a powerlaw bound in $d>2$.*

We next have to consider a general configuration of points $x_i = (\tau_i + i t_i, \mathbf{x}_i)$. One may suspect that, for $\tau_1 > \tau_2 > \tau_3 > \tau_4$, the same strategy will work, namely that we will have $|\rho|, |\bar{\rho}| < 1$. In fact this turns out to be true, but somewhat difficult to prove directly, since $z, \bar{z}$ do not have simple expressions in terms of x_i beyond 2d. There is one case when $z, \bar{z}$ do have explicit expressions: for reflection positive configurations, which can be suitably generalized to complex times. We will use this fact, and reduce the general case to the reflection positive one.

Unlike in what we have done so far, it will be important not just to use explicit formulas for 4pt function, but to recall their OPE origin. We have the OPE:

$$\varphi(x_1)\varphi(x_2) = \sum_{\mathcal{O}} \sum_I c_I^{\mathcal{O}}(x_1, x_2, y)\mathcal{O}_I(y) \tag{110}$$

where the first sum is over primaries in the OPE, and the second sum is over all states in the multiplet of $\mathcal{O}$. We can think of the index I as a list of tensor indices, some coming from $\mathcal{O}$ itself if it was a tensor, others introduced by derivatives.

We are interested in 4pt function $\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle$ where two points have positive x^0 time and two negative: $\tau_1, \tau_2 > 0 > \tau_3, \tau_4$. Let us use the OPE for the first two points with $y = N = (1, \vec{0})$ the "North pole" , while for the second pair with $y = S = (-1, \vec{0})$ the "South pole". We have

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = \sum_{\mathcal{O}} c_I^{\mathcal{O}}(x_1, x_2, N) c_J^{\mathcal{O}}(x_3, x_4, S) G_0^{IJ} \tag{111}$$

where we defined the Gram matrix:

$$G_0^{IJ} = \langle\mathcal{O}_I(N)\mathcal{O}_J(S)\rangle \tag{112}$$

In a reflection-positive theory the matrix $G^{IJ} = G_0^{IJ}(-1)^{N_J}$ is a real, positive-definite matrix, where N_J is the number of τ's in J.[16]

Let us define operation θ by $x^0 \to -x^0$. We claim that $c_I^{\mathcal{O}}$ transforms under θ as:

$$c_I^{\mathcal{O}}(x_1^\theta, x_2^\theta, y^\theta) = c_I^{\mathcal{O}}(x_1, x_2, y)(-1)^{N_I}. \tag{113}$$

This is just a property of rotation-invariant tensors. Indeed c_I is some tensor built out of components of x_1, x_2, y and of $\delta_{\mu\nu}$. When x^0 components of vectors flip sign, the x^0 components of $c_I^{\mathcal{O}}$ coming from vectors do so as well, and components δ_{00} are unaffected which is consistent with the above rule because $(-1)^2 = 1$. Scalars like $x_1 \cdot x_2$ are invariant.[17]

Applying the rule (113) in (111), we get:

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = \sum_{\mathcal{O}} c_I^{\mathcal{O}}(x_1, x_2, N) c_J^{\mathcal{O}}(x_3^\theta, x_4^\theta, N) G^{IJ} \tag{114}$$

which now involves the positive-definite matrix G^{IJ}.

16. We are assuming that real basis of operators has been chosen.

17. It is important for this discussion that $\mathcal{O}$'s which occur in the OPE of two scalars are symmetric traceless scalars, and thus thd ε-tensor does not appear. In presence of the ε-tensor there would be an extra minus sign in the transformation of c_I.

We claim that the r.h.s. of (114) converges for all x_1, x_2 at $\tau > 0$ and $x_3, x_4 < 0$. This may be surprising because we may not necessarily separate all such pairs of points with spheres centered at N, S. So it looks like the most naive condition for the OPE convergence is not always satisfied. To go around this apparent difficulty, let us do the conformal map under which N, S are mapped to $0, \infty$. In this radial quantization frame the images of x_1, x_2 will be inside the sphere of radius 1, while those of x_3, x_4 outside. So in the radial quantization frame the OPE converges. When comparing the OPE expansions in both frames, we should keep in mind that a single derivative $\partial^\alpha \mathcal{O}(0)$ is mapped into a linear combination of derivatives $\partial^\beta \mathcal{O}(N)$ with $|\beta| \leqslant |\alpha|$. However, if we sum over all descendants up to order n, and then take $n \to \infty$, then partial sums of two OPE expansions are the same. In what follows we will understand the summation in (114) in this sense, which is guaranteed to converge.

Now let us set $x_3 = x_2^\theta$, $x_4 = x_1^\theta$ in (114). Then we have:

$$\langle \varphi(x_1)\varphi(x_2)\varphi(x_2^\theta)\varphi(x_1^\theta)\rangle = \sum_{\mathcal{O}} c_I^{\mathcal{O}}(x_1, x_2, N) c_J^{\mathcal{O}}(x_1, x_2, N) G^{IJ} \tag{115}$$

This has to be positive in an RP theory. Positive-definite matrix G^{IJ} satisfies $\xi_I \xi_J^* G^{IJ} \geqslant 0$ while we have $c_I c_J G^{IJ}$. Below we will argue that $c_I(x_1, x_2, N)$ are real for Euclidean $x_{1,} x_2$ in an RP theory. Thus (115) is positive as required.

If we use conformal invariance, all coefficients $c_I^{\mathcal{O}}$ are equal to a single number, OPE coefficient $C_{\varphi\varphi\mathcal{O}}$, times a real tensor structure. Let us show that $C_{\varphi\varphi\mathcal{O}}$ must be real in an RP theory. For this consider a state which is a mix of a two-point and a one-point insertions at $\tau > 0$:

$$\langle \mathcal{O}_K(N)| + \varepsilon \langle \varphi(x_1)\varphi(x_2)|, \tag{116}$$

for some index K and $\varepsilon \in \mathbb{C}$. In a reflection positive theory the following expression has to be real (as well as positive):

$$\langle O_K(N)\mathcal{O}_K(S)\rangle(-1)^{N_K} + \varepsilon\langle \mathcal{O}_K(N)\varphi(x_1^\theta)\varphi(x_2^\theta)\rangle + \varepsilon^*\langle \varphi(x_1)\varphi(x_2)(-1)^{N_K}\mathcal{O}_K(S)\rangle + \varepsilon\varepsilon^*\langle \varphi(x_1)\varphi(x_2)\varphi(x_1^\theta)\varphi(x_2^\theta)\rangle. \tag{117}$$

Now consider the limit $\varepsilon \to 0$ and impose the reality. The order ε terms sum up to $(\varepsilon + \varepsilon^*) C_{\varphi\varphi\mathcal{O}}$ times the same tensor structure. So $C_{\varphi\varphi\mathcal{O}}$ must be real.

Up to now we considered only Euclidean, real, configurations of points, but now we wish to consider analytic continuation $x^0 \to \tau + it$ (keeping spatial coordinates real). We keep reflection operation θ as before $x^0 \to -x^0$, but as we will see a particularly nice operation will be $*\theta$ under which $\tau \to -\tau$, $t \to t$.

Analytic continuation may be defined by taking the OPE (110) and analytically continuing: $x_i^0 \to \tau_i + it_i$. Then we use the analytically continued OPE in (111), which defines an analytic function of x_i^0, provided that it converges that we need to check. Of course analytically continued c_I's are now complex. Eq. (113) remains true after analytic continuation, and this (114). Finally since c_I is real before continuation, after continuation we have

$$c_I^{\mathcal{O}}(x_1, x_2, N)^* = c_I^{\mathcal{O}}(x_1^*, x_2^*, N). \tag{118}$$

This allows us to rewrite (114) as

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle=\sum_{\mathcal{O}} c_I^{\mathcal{O}}(x_1,x_2,N)c_J^{\mathcal{O}}(x_3^{*\theta},x_4^{*\theta},N)^*\,G^{IJ} \tag{119}$$

Now observe that by setting $x_3=x_1^{*\theta}, x_4=x_2^{*\theta}$ we get

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_1^{*\theta})\varphi(x_2^{*\theta})\rangle=\sum_{\mathcal{O}} c_I^{\mathcal{O}}(x_1,x_2,N)c_J^{\mathcal{O}}(x_1,x_2,N)^*\,G^{IJ} \tag{120}$$

The r.h.s. of this equation remains real and positive even for complex times. We conclude that the 4pt functions at complex times in $*\theta$-symmetric configurations are real and positive.[18] [19]

Notice also that as a consequence of the last two equations we have Cauchy-Schwarz inequality for the 4pt function analytically continued to complex times:

$$|G(x_1,x_2,x_3,x_4)|\leqslant G(x_1,x_2,x_2^{*\theta},x_1^{*\theta})G(x_4^{*\theta},x_3^{*\theta},x_3,x_4). \tag{121}$$

We are thus reduced to study the 4pt functions at complex times in $*\theta$-symmetric configurations.

To prove a bound on $G(x_1,x_2,x_2^{*\theta},x_1^{*\theta})$ we find the corresponding $z,\bar{z}$. We expect that they will be real and in $(0,1)$, since then 4pt functions will be real and positive. By constant shifts in t and $\mathbf{x}$, it's enough to consider the configuration

$$x_1=(\tau_1+it,\mathbf{x}), x_2=(\tau_2,0), x_3=x_2^{*\theta}=(-\tau_2,0), x_4=x_1^{*\theta}=(-\tau_1+it,\mathbf{x}) \tag{122}$$

Then a simple computation gives:

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}=\frac{[((\tau_1-\tau_2)^2+it)^2+\mathbf{x}^2]\times c.c.}{\tau_1-\tau_2\to\tau_1+\tau_2}=\frac{[(\tau_1-\tau_2)^2-t^2+\mathbf{x}^2]^2+4(\tau_1-\tau_2)^2t^2}{\tau_1-\tau_2\to\tau_1+\tau_2}$$
$$=\frac{[(\tau_1-\tau_2)^2+t^2+\mathbf{x}^2]^2-4t^2\mathbf{x}^2}{\tau_1-\tau_2\to\tau_1+\tau_2}=\frac{[(\tau_1-\tau_2)^2+(t-|\mathbf{x}|)^2][(\tau_1-\tau_2)^2+(t+|\mathbf{x}|)^2]}{[(\tau_1+\tau_2)^2+(t-|\mathbf{x}|)^2][(\tau_1+\tau_2)^2+(t+|\mathbf{x}|)^2]} \tag{123}$$

while

$$v=\frac{x_{23}^2x_{14}^2}{x_{13}^2x_{24}^2}=\frac{4\tau_2^2\times4\tau_1^2}{[(\tau_1+\tau_2)^2+(t-|\mathbf{x}|)^2][(\tau_1+\tau_2)+(t+|\mathbf{x}|)^2]} \tag{124}$$

From here we realize that $u=z\,\bar{z}, v=(1-z)(1-\bar{z})$ is solved by

$$z=\frac{(\tau_1-\tau_2)^2+(t-|\mathbf{x}|)^2}{(\tau_1+\tau_2)^2+(t-|\mathbf{x}|)^2},\qquad \bar{z}=\frac{(\tau_1-\tau_2)^2+(t+|\mathbf{x}|)^2}{(\tau_1+\tau_2)^2+(t+|\mathbf{x}|)^2} \tag{125}$$

18. For completeness let us argue that the expansion in the r.h.s. of (120) converges for complex times. Denote by S_n its partial sums with the cutoff described above, summing over all descendants up to order n for each primary. We have to appeal to the kinematic fact that these partials sums, for complex as for real times, can be obtained from the l.h.s. of (120) by taking z, $\bar{z}$ expansion and cutting off to $h+\bar{h}\leqslant\Delta+n$ in each conformal block. The latter expansion consists of positive terms and converges for complex times, since it's term by term bounded by the same expansion at real times, whose convergence is our basic assumption (finiteness of Euclidean 4pt functions).

19. This should not be surprising in view of the familiar QFT relation: $\varphi(\tau+it)^\dagger=\varphi(-\tau+it)$ which follows from $\varphi(x^0)=e^{iHx^0}\varphi(0)e^{-iHx^0}$. The point is that we don't want to use this relation which assumes that φ and H have been realized as Hermitean operators acting on the same Hilbert space. In CFT we have OPE which allows us to recover the same property in a different way.

As promised we have $0 < z, \bar{z} < 1$, both real.

Polynomial bound follows.

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