Supersymmetry in Disorder and Chaos

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Contents

1 Lecture 1

1.1 Introduction

Let the Hamiltonian of a particle be

$$H_r = \epsilon(-i\nabla_r) + U_r \tag{1}$$

Here ϵ is a function, for instance $\epsilon(p) = \frac{p^2}{2m} - \epsilon_F$ where ϵ_F is the Fermi energy. And U_r is a random potential. As we are looking for universal results, we may as well assume that U_r has a Gaussian law

$$\langle U_r \rangle = 0 \quad , \quad \langle U_r U_s \rangle = \gamma \delta_{rs} \tag{2}$$

Classically, if the disorder is weak, i.e. U_r is smaller than the energy of particles, then the conductivity behaves as

$$\sigma = \frac{e^2 n \tau}{m} \tag{3}$$

where n is the density and τ the scattering time (related to the mean free path ℓ by $\ell = \tau v_F$ where v_F is the Fermi velocity, and to the constant γ by $\gamma = \frac{1}{2\pi\nu\tau}$ where ν is the density of states). This formula is not dependent on the lattice spacing a. This formula will also describe the behaviour of quantum systems in situations where the quantum coherence is destroyed, for instance by heating the sample. But we will assume the temperature to be zero in the following, i.e. we neglect inelastic scattering.

According to Anderson, if τ becomes small enough so that $\ell \sim a$, then we have localization and $\sigma = 0$ in contradiction with the classical formula. So σ will vanish if γ exceeds a critical value γ_c. The transition to $\sigma = 0$ is continuous according to Anderson, and discontinuous according to a scenario by Mott, which is now disbelieved.

In our system we can have extended states with wavefunctions of the type e^{ikr}, and localized states. Energy levels of localized states differ by $\Delta E \sim e^{-L}$ while for extended states we have $\Delta E \sim \frac{1}{L}$, where L is the size of the sample. In the presence of disorder, two localized eigenstates with very close energies may well be localized at very different positions.

1.2 Renormalization group argument

Anderson localization is the statement that in 3d, if γ is large enough, then states will be localized. Mott has shown that in an 1d disordered system all states are localized. The difficult case of 2d was studied in 1979 by Abrahams, Anderson, Licciardello and Ramakrishnan, who showed that all states are localized. They studied the conductance $g = \sigma L^{d-2}$, and its behaviour under renormalization. (A more accurate description shoud not use just conductances, but distributions of conductances, and their momenta $\langle g \rangle$, $\langle g^2 \rangle$, etc.) The basic assumption is that its behaviour under a rescaling by a factor b is $g(bL) = f(b, g(L))$ for some function f, which leads for values close to $b = 1$ to the equation

$$\frac{d \log g(L)}{d \log L} = \beta(g(L)) \tag{4}$$

for some function β. If the conductance is large, we expect that it behaves as $g = g_0 L^{d-2}$ so that

$$\lim_{g\to\infty} \beta(g) = d - 2 \tag{5}$$

On the other hand, localization occurs for small g. In the presence of localization, the conductance decreases exponentially with sample size, $g = g_0 e^{-\alpha L}$, and we have

$$\lim_{g\to 0} \beta(g) = \log \frac{g}{g_0} \tag{6}$$

Conversely, if we know β we can detect localization by just looking as its sign, with $\beta \leq 0$ indicating localization, as it means larger samples have lower conductances.

The next assumption is that $\beta(g)$ is a monotonously increasing function. (This assumption can be violated in the presence of spin-orbit impurities.) This assumption allows us to determine the sign of β. If $d \leq 2$ then $\beta \leq 0$ and we have localization. On the other hand, if $d = 3$, then we have an unstable renormalization group fixed point g_c such that $\beta(g_c) = 0$.

1.3 Relation with random walks

It is not a coincidence that the critical dimension $d = 2$ for localization coincides with the critical dimension of a random walk. Consider indeed the probability P_R that a random walker visits again his point of origin. To compute P_R, remember that the probability to be at distance r after a time t is

$$P(r,t) = \frac{1}{(4\pi D t)^{\frac{d}{2}}} e^{-\frac{r^2}{Dt}} \tag{7}$$

Then we have

$$P_R \sim \int_{t_0}^{t} P(0,t) dt \tag{8}$$

which diverges at large t for $d \leq 2$, indicating an almost certain return to the origin.

A quantum diffusion process will differ from classical diffusion if pronounced interferences occur. A possible source of such interferences is a path

which crosses itself. This is because the loop which is part of such a path can be followed in two possible directions, which however produce the same phase for the amplitude $\psi = e^{i\frac{pr}{\hbar}}$, and interfere constructively. It is possible to detect this effect experimentally, as it can be destroyed by adding a magnetic field. In the presence of such a field the phase around a loop if $\varphi = \oint (p - \frac{e}{c}A)dr$, and the phase difference between two orientations of the same loop is $\Delta\varphi = 2\pi\frac{\phi}{\phi_0}$ where ϕ is the magnetic flux in the loop. This leads to the phenomenon of negative magnetic resistance, i.e. the growth of conductivity when a magnetic field is applied. This is because conductivity is diminished by localization, and our constructive interferences tend to produce localization (a phenomenon called weak localization). Indeed, these interferences lead to quantum corrections to σ, and the first quantum correction diverges if $d = 2$.

Of course, the interferences can be destroyed in the presence not only of a magnetic field, but also of temperature, as nonzero T implies a cutoff (called decoherence time) in the integral (8) over t. Actually, the magnetic field destroys the interferences in the first quantum correction, but not necessarily in the following corrections, whereas nonzero T destroys the interferences at all orders.

2 Lecture 2: Perturbation theory

We want to treat the Hamiltonian (1) perturbatively, where of course the random potential U_r will be the perturbation. We are looking for universal results, so we assume the law of U_r to be Gaussian as in eq. (2). Of course, the results will be different for different universality classes specified by the symmetries that are broken by the impurities or external fields: for example, a magnetic field or magnetic impurities break the time reversal invariance while spin-orbit impurities break the spin rotation symmetry.[1]

2.1 Green's functions

The basic function which characterizes the system is the Green's function, which comes in two varieties: retarded (R) or advanced (A), corresponding

[1]This chapter has been written by S. Ribault and H. Meier

to two independent solutions of the second-order equation

$$(\epsilon - H)G^{R/A}(\epsilon, r, r') = \delta(r - r'). \tag{9}$$

(If the equation was first-order, its solution would be simple, which would be a problem for the employment of theoretical physicists.) The solutions can be written in terms of eigenfunctions ϕ_n,

$$H\phi_n = \epsilon_n \phi_n \Rightarrow G^{R/A}(\epsilon, r, r') = \sum_n \frac{\phi_n(r)\phi_n^*(r')}{\epsilon - \epsilon_n \pm i\delta}. \tag{10}$$

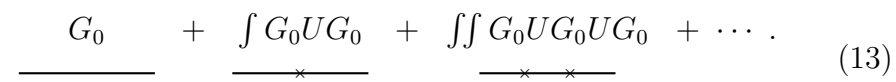
Here δ is an infinitesimal positive number. It is easy to compute the Green's function if we neglect the random potential and keep only the term $H = H_0 = \epsilon(-i\nabla)$. After a Fourier transformation, we find the function of the momenta p, p':

$$G_0^{R/A}(\epsilon; p, p') = \delta(p + p')G_0^{R/A}(\epsilon, p) \quad \text{with} \quad G_0^{R/A}(\epsilon, p) = \frac{1}{\epsilon - \epsilon(p) \pm i\delta}. \tag{11}$$

The full Green's function G can be computed recursively from the leading-order term G_0 and the potential U_r as

$$G(r, r') = G_0(r, r') - \int G_0(r, r'')U_{r''}G(r'', r')dr'' \tag{12}$$

where we omit the variable ϵ and the index $\{R/A\}$, which are spectators in this formula. Schematically, the Green's function can be represented as

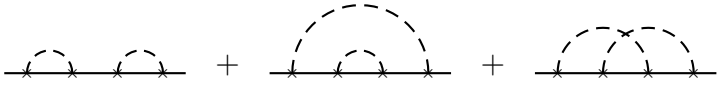
G =  (13)

2.2 Averaging over disorder

Let us compute averaged Green's functions. Since the law (2) of the potential U_r is Gaussian, only even terms in the expansion (13) are nonzero. Averaging is done diagramatically by connecting pairs of crosses (i.e. the insertions of U_r) with so-called impurity lines. For instance, at order two, there is only one diagram,

 . (14)

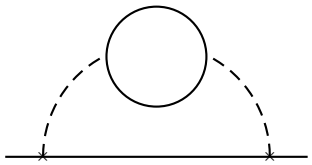
At order four, we have to consider a priori three diagrams,



(15)

In dimensions $d \geq 2$, the third diagram, which contains crossed impurity lines, is of order $1/(\tau\epsilon_0)$.[2] If the scattering time τ is large compared to the inverse of the Fermi energy ϵ_0 or: if $1/(\tau\epsilon_0) \ll 1$, the third diagram is negligible. In general: All diagrams that contain crossed impurities may in the leading order in $1/(\tau\epsilon_0)$ be neglected ("non-crossing approximation"). Note, however, that in one dimension, this approximation is not justified and localization is observed while in dimensions $d \geq 2$ the smallness in $1/(\tau\epsilon_0)$ will prevent localization to show up in this approach.

This diagrammatic exercise can be understood as a first hint of the interest of supersymmetry for such systems. This is because our diagrams cannot contain any loops of the type



(16)

If we wanted to interpret our diagrams as Feynman diagrams of some effective field theory with the impurity lines representing the interaction between the fields, such loop diagrams would in general contribute. They would however be absent in the special case of a supersymmetric theory. In such a theory, we will not only have the same numbers of bosons and fermions, so that we have bosonic and fermionic loops with opposite signs, but we also have the same interactions for bosons and fermions so that eventually the bosonic and fermionic loops cancel each other.

2.3 Detecting localization

In order to study localization and to show universality (i.e. the independence of the results from the details of the spectrum $\epsilon(p)$), we will have to compute fairly complicated correlation functions. As a warm-up exercise, let us start with the average density of states, a quantity that however does not allow to detect localization.

[2]Note that the Planck constant $\hbar$ has been set to one.

The disorder-averaged density of states at the point r is given by

$$\langle \rho(\epsilon, r) \rangle = \left\langle \sum_n \phi_n(r)\phi_n^*(r)\delta(\epsilon - \epsilon_n) \right\rangle = -\frac{1}{\pi}\Im[\langle G^R(\epsilon, r, r)\rangle]$$
$$= -\frac{1}{2\pi i}\left\langle G^R(\epsilon, r, r) - G^A(\epsilon, r, r)\right\rangle \tag{17}$$

(After averaging, we expect this quantity to be independent from r.) We thus need to find the disorder-averaged Green's functions by summing the series of all non-crossing diagrams that begins with the diagrams in (14) and (15). Equivalently, we can calculate the self-energy due to diagram (14) in a self-consistent way [3] and arrive at the Green's function in the self-consistent Born approximation,

$$\left\langle G^{R/A}(\epsilon, p)\right\rangle = \frac{1}{\epsilon - \epsilon(p) \pm \frac{i}{2\tau}}. \tag{18}$$

As expected, $G_0^{R/A}$ is diagonal in momentum space as the translational symmetry has been restored by the disorder-averaging.

In order to calculate the density of states $\langle \rho(\epsilon, r)\rangle$, eq. (17), it remains to perform a last momentum integration to find the Green's function at coinciding points. Doing so, we find

$$\langle \rho(\epsilon, r)\rangle = -\frac{1}{\pi}\Im\Big[\int \frac{d^d p}{(2\pi)^d}\left\langle G^R(\epsilon, p)\right\rangle\Big] = -\frac{1}{\pi}\Im\Big[\nu\int \frac{d\xi}{\epsilon - \xi + \frac{i}{2\tau}}\Big] = \nu. \tag{19}$$

This result coincides with the density of states in the absence of disorder and thus the density of states does — as expected — not allow to detect effects of localization.

If we wish to detect localization effects, we need to consider less trivial correlation functions. Among the functions of physical relevance, there are the correlation functions

$$X^{ab}(r, r', t) = i\theta(-t)\left\langle \Big[j^a(r, t), j^b(r', 0)\Big]\right\rangle \tag{20}$$

[3] In these calculations, integrals over the momentum reduce as usual to integrals over $\xi = \epsilon(p) - \epsilon_0$ while transforming the measure as $d^d p/(2\pi)^d = \nu d\xi$ with ν being the density of states for the spectrum $\epsilon(p)$ at the Fermi edge.

where $\theta(t)$ is the step function[4] and the j^a denotes for $a = 0$ the operator for the charge density,

$$j^0(r) = e \sum_{kk',\sigma} \phi_k^*(r)\phi_{k'}(r) a_{k\sigma}^\dagger a_{k'\sigma}, \tag{21}$$

and for $\alpha = 1, 2, 3$ the operator for the current density,

$$j^\alpha(r) = \sum_{kk',\sigma} \Big\{ \frac{ie}{2m} \Big(\big[\nabla^\alpha \phi_k^*(r)\big] \phi_{k'}(r) - \phi_k^*(r) \nabla^\alpha \phi_{k'}(r) \Big) - \frac{e^2}{mc} A^\alpha \phi_k^*(r) \phi_{k'}(r) \Big\} a_{k\sigma}^\dagger a_{k'\sigma}. \tag{22}$$

$a_{k\sigma}^\dagger$ and $a_{k'\sigma}$ are election creation and annihilation operators for the state k.

The angular brackets in eq. (20) stand for quantum-thermal averaging. Clearly, when calculating a physical quantity, one should finally average also over disorder. The correlation function X^{00} is the density-density correlation function while the components $X^{\alpha\beta}$ are related to the conductivity of the sample.

Another quantity of interest is the level-level-correlation function,

$$R(\omega) = \left\langle \frac{1}{4\omega\nu^2 V^2} \sum_{kl} \big[n(\epsilon_k) - n(\epsilon_l) \big] \delta(\omega - \epsilon_m + \epsilon_k) \right\rangle. \tag{23}$$

Herein, $n(\epsilon)$ is the Fermi distribution function.

Rewriting either the correlation functions of eq. (20) or the one in eq. (23) in terms of retarded and advanced Green's function, one finds expressions of the form

$$Y(\epsilon, \omega, r, r') = G^R(\epsilon - \omega, r', r) \left[G^A(\epsilon, r, r') - G^R(\epsilon, r, r') \right]. \tag{24}$$

For instance,

$$X^{00}(r, r', t) = \frac{e^2}{(2\pi)^2 i} \sum_{\sigma\sigma'} \int \exp\big\{ -i\omega t \big\} n(\epsilon) \times \big[Y^{00}(\epsilon, \omega, r, r') - Y^{00}(\epsilon, -\omega, r, r') \big] d\epsilon d\omega. \tag{25}$$

[4] In this lecture, we use the convention that

$$\theta(t) = \begin{cases} 0 & , \quad t > 0 \\ 1 & , \quad t < 0 \end{cases}.$$

In the case of a localized state, the density-density correlation function will have a simple dependence on ω,

$$Y(\epsilon,\omega,r,r') \underset{localized}{=} \frac{2\pi}{-i\omega}\sum_n |\phi_n(r)|^2|\phi_n(r')|^2\delta(\epsilon-\epsilon_n) \tag{26}$$

On the other hand, in the case of non-localized states (for example a metal), we have in Fourier representation

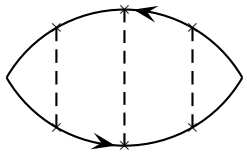
$$Y(\epsilon,\omega,k) \underset{non-localized}{=} \frac{2\pi}{D_0k^2-i\omega}. \tag{27}$$

A localized state is therefore characterized by the vanishing of the diffusion coefficient D_0.

To find eq. (27), we have to compute

$$\langle Y(\epsilon,\omega,k)\rangle = \int dp\ \left\langle G^R(\epsilon-\omega,p)G^A(\epsilon,p+k)\right\rangle \tag{28}$$

where we neglect the G^RG^R term, which yields a trivial contribution. Diagrammatically, we have to take into account besides the impurity lines connecting G^R with itself and G^A with itself those that connect G^R with G^A. In the non-crossing approximation, this amounts to summing over "ladder" diagrams of the type



. (29)

Diagrams where impurity lines connect a Green's function with itself are taken into account by the self-energy term in the Green's function as in eq. (18).

Each "rung" in (29) yields the factor

$$\Pi(k,\omega) = \int \frac{d^dp}{(2\pi)^d}\langle G^R(\epsilon,p)\rangle\langle G^A(\epsilon-\omega,p-k)\rangle \tag{30}$$

and then the whole ladder series

$$\langle Y(\epsilon,\omega,k) = \frac{\Pi(k,\omega)}{1-(2\pi\nu\tau)^{-1}\Pi(k,\omega)}. \tag{31}$$

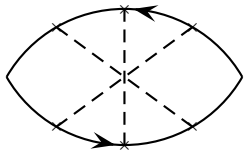
For small ω and k, one obtains eq. (27).

For the conductivity, there are no corrections due to the ladder and from the integral $\int \langle G^R(\epsilon-\omega,p)\rangle\langle G^A(\epsilon,p)\rangle(d^dp/(2\pi^d))$, we find

$$\sigma(\omega) = \frac{\sigma_0}{1-i\omega} \tag{32}$$

where $\sigma_0 = 2e^2\nu D_0$ with the diffusion coefficient $D_0 = v_0^2\tau/d$ is the classical Drude conductivity.

While such calculations can be useful for certain systems (like superconductors), they fail to show localization. In 1979, after Anderson et al. conjectured localization using the renormalization group argument, Gorkov, Larkin and Khmenitskii found what perturbation theory missed: These are the diagrams where impurity lines cross *maximally*. While individually negligible, the resummation of these diagrams results in a significant contribution. Localization corrections are indeed found by taking into account the diagrams of the type



. (33)

Ladder diagrams with no crossed impurity lines like  describe (classical) diffusive motion and thus the mode obtained by resumming the series is called "diffuson". The maximally crossed diagrams can be rewritten in the form  . The mode due to these ladders are called "cooperon" propagators. The evaluation of the cooperon loops is analogous to those loops appearing in the diffusons. The cooperons correspond to diffusive motion along those paths which produced constructive interferences in the quantum diffusion process of Section 1.3. They lead to corrections to the conductivity σ of the type

$$\frac{\sigma}{\sigma_0} = 1 - \frac{1}{\pi\nu}\int \frac{1}{Dq^2 - i\omega}\frac{d^dq}{(2\pi)^d} \tag{34}$$

which are infrared-divergent if $d \leq 2$, showing localization.

Since it is clearly not satisfactory to just see localization effects in terms of some divergences, we will want to actually sum the corrections to σ in a

systematic way. This is what the supersymmetric effective field theory will be derived for.